Regularity Aspects for Combinatorial Simplicial Surfaces

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1 Introduction

1.1 Motivation

Combinatorial surfaces capture essential properties of continuous surfaces (like spheres and tori) in a discrete manner that lends itself more easily to a computational approach. One way to obtain a combinatorial surface is by subdividing a continuous surface into triangles, like in Figure 1.1.

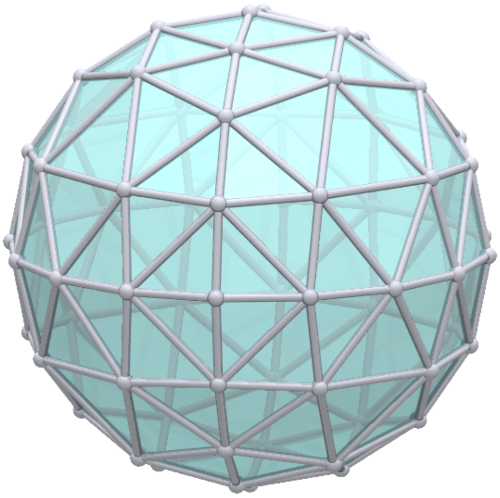


Figure 1.1: Combinatorial surface ([53])

The shift from continuous surfaces to combinatorial ones has been very successful in several areas of mathematics:

- In Algebraic Topology, the concept of *homology* was introduced to formalise the notion of a "hole". It is difficult to compute the homology for continuous surface directly, but very easy for combinatorial ones. Thus, a basic approach is replacing a continuous surface by a combinatorial one with the same homology. Then, homology can be computed efficiently.

- Physics commonly deals with continuous surfaces, e. g. particle movement in a surface according to a differential equation, or movement of the whole surface under pressure.

Often, an analytical solution is not feasible or impractical, so one turns to numerical ones. In numerical mathematics, continuous surfaces are replaced by combinatorial ones, where the solution is feasible. Of course, solutions for the continuous and the combinatorial surface differ, but this error can be controlled if the combinatorial surface is a "good" approximation to the continuous one.

- In some cases, combinatorial surfaces are the most natural model. Consider paper folding with several well–defined crease lines. Apart from those, the paper is flat. Here, it is most natural to consider a crease line not as a set of many points, but rather as a single element. We have shown in [10] that a combinatorial description is sufficient to describe many properties of folding. For example, many "impossible folds" are already impossible on the combinatorial level.

In all these examples, it is crucial that the combinatorial properties "preserve" aspects of the continuous surface to pull back the computational results onto the continuous case. Since the combinatorial properties are central, it makes sense to study them for their own sake.

1.2 Main projects

The theoretical content of this thesis focusses on *regularity aspects*. Combinatorial surfaces are sometimes difficult to work with since they only have a very weak structure (usually some kind of incidence relation). To obtain theoretical results, one often has to impose further assumptions. A very strong one is *regularity*, i. e. each vertex is incident to the same number of faces.

In this thesis, we transfer some results for regular surfaces to non–regular ones. There are three main theoretical results: A differential–geometric perspective on nets of combinatorial surfaces, the construction of global invariants with an infinite regular extension, and the characterisation of geodesic self–dual regular surfaces. The fourth main result is of a more practical nature, designing and implementing the GAP–package `SimplicialSurfaces` ([13]) to perform computations with combinatorial surfaces. Most concepts presented in this thesis are contained in the package.

1.2.1 Surface nets (Chapter 7)

Embeddings of combinatorial surfaces into some $\mathbb{R}^n$ (like in Figure 1.1) are often of interest, in particular when the surface is constructed purely combinatorially. For simplicial complexes, an embedding can be easily constructed if the dimension of the space is large enough. For smaller dimensions (especially for $\mathbb{R}^3$, like in [16]) the problem becomes much harder. We refer to [55] for an overview.

Thus, one is inclined to look at weaker notions. One of these is the representation as a *net* in $\mathbb{R}^2$. For the octahedron, we have the net depicted in Figure 1.2. Clearly, some of the labels appear multiple times. Similar to a net of the cube, these denote the same vertex in the embedding into $\mathbb{R}^3$.

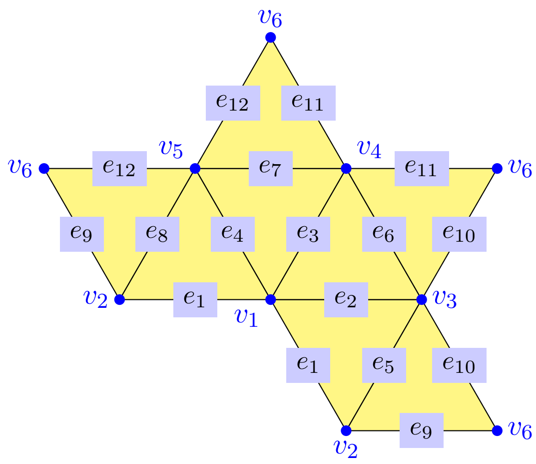


Figure 1.2: Net of an octahedron

Although nets are commonly used in the depiction of cell complexes (compare [36, page 5]), properties of these nets are rarely brought into correspondence with properties of the combinatorial surface. One project of this thesis is to focus on this correspondence.

If we only consider nets with equilateral triangles, all of these triangles lie in the infinite combinatorial surface induced by the hexagonal lattice (compare Figure 1.3). Additionally, for each pair of edges with the same label, we store a symmetry of the

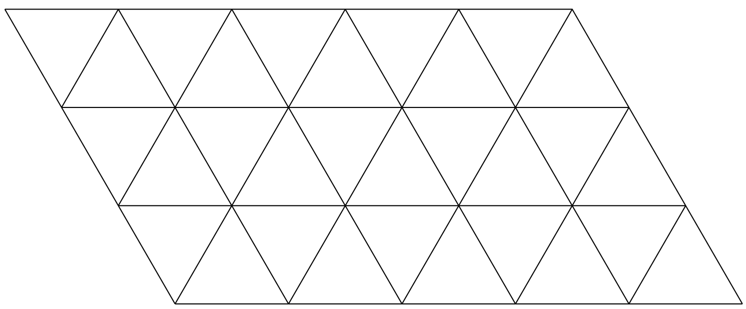


Figure 1.3: Excerpt from the hexagonal lattice surface

hexagonal lattice mapping one of them to the other. We call these maps *transition maps*, similar to the coordinate transition maps in differential geometry.

The infinite hexagonal lattice, interpreted as a combinatorial surface, is regular and has several nice properties: it is orientable and allows several "nice" edge–colourings. Theorem 7.3.14 in Section 7.3 shows that a general combinatorial surface has one of these properties if and only if it has a net whose transition maps preserve the properties in the infinite surface.

In summary, we related properties of combinatorial surfaces with possible shapes of their nets, providing a good foundation for further research about these correspondences.

1.2.2 Surface modifications (Chapter 8)

Many combinatorial surfaces look quite similar and can be obtained from each other by a sequence of rather simple changes (like changing an edge). We can exploit these similarities by transferring structural understanding from one surface to the other. In [27, Section 3.4], the modification "edge flip" is used to enumerate all triangulations, and [39] gives an upper limit for the number of those modifications. Several different modifications (that also preserve colouring) are presented in [40], together with several interesting theoretical applications.

Another often studied modification is the *vertex split* ([35, D22, Subsection 7.8.3]), illustrated in Figure 1.4. It can be used to construct large spherical triangulations from the tetrahedron ([57] and [65]). In this thesis, we focus on their impact on combinatorial

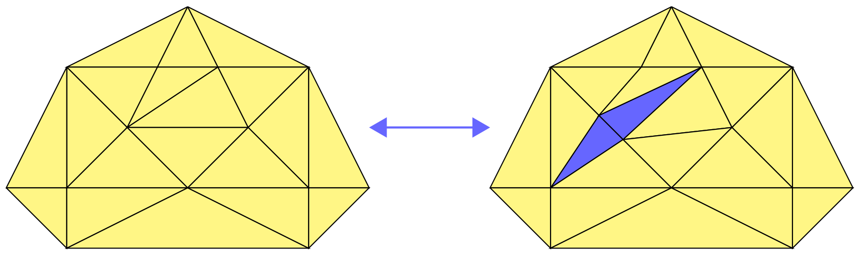


Figure 1.4: Modification of a combinatorial surface

surfaces with a single boundary like the one illustrated in Figure 1.5. Given two of these, we would like to know whether we can construct one from the other by using vertex splits. In particular, we are interested in *global invariants*, i. e. properties that

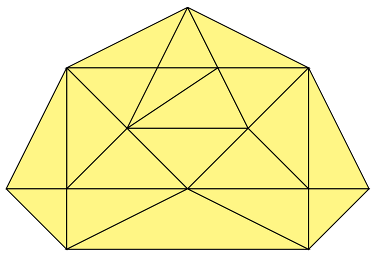


Figure 1.5: Combinatorial surface with a single boundary

stay invariant under these modifications.

In Chapter 8, we construct several of these invariants by using the *infinite regular extension* (Theorem 8.3.9 and Theorem 8.3.24). Conceptually, we extend the combinatorial surface along its boundary, such that all newly constructed vertices are incident to exactly six faces.

Two combinatorial surfaces with different global invariants cannot be obtained from each other by vertex splits. Thus, future work has to focus one combinatorial surfaces with the same global invariants. Fixing these invariants gives the surfaces more structure, so this seems to be a feasible approach.

1.2.3 Highly symmetric surfaces (Chapter 9)

Starting with the platonic solids, highly symmetric surfaces are often studied ([25]). They are usually easier to work with than general combinatorial surfaces, which allows construction and understanding of rather large examples. In addition, they often appear in applications satisfying certain symmetries. Thus, classifying highly symmetric surfaces is a worthwile endeavour.

Combinatorial surfaces where every vertex is incident to the same number of faces are studied in [23] and [22]. But there are other symmetries to consider, like the duality of swapping the roles of vertices and faces of a combinatorial surface. Surfaces that are unchanged under this duality satisfy an *external surface symmetry*. The self–dual spherical surfaces are classified in [4]. Actually, there are more external surface symmetries, as shown in [71]. Naturally, there have been classifications of those surfaces that are invariant under all of these operations ([3]).

In this thesis, we focus on self–duality with respect to a particular external surface symmetry in [71], which we call *geodesic duality*. Theorem 9.6.1 in Chapter 9 characterises the regular combinatorial surfaces that are unchanged under geodesic duality.

To do so, we identify the regular surfaces with certain subgroups of triangle groups, and expand this correspondence to encompass geodesic self–duality.

Our characterisation is successful, but we do not achieve a full classification. In particular in the infinite case, further research is necessary. This classification was also submitted in [11].

1.2.4 Software implementation (Chapter 10)

Theoretical understanding does not arise from a vacuum. Depending on the field of study, different aspects are more or less pronounced. In the field of combinatorial geometry, the direct study of examples is a central aspect, both to generate and to test hypotheses.

Unfortunately, combinatorial surfaces can be difficult to handle on paper, as an incidence geometry almost has to be written down relation by relation. Fortunately, this work is cut out for a computer. Therefore, it is a natural choice to use software to study combinatorial surfaces. For that reason, part of this thesis is writing a `GAP`–package ([33]) to handle computations with combinatorial surfaces.

The package `SimplicialSurfaces` ([13]) encodes combinatorial surfaces and several common algorithms efficiently, allowing the user to focus fully on the underlying mathematical structure. Notable features include a library of surfaces that greatly facilitates the testing of conjectures, and a flexible framework to build custom code for combinatorial surfaces, allowing for a wide variety of different research. Chapter 10 gives an overview of the package.

In total, the package is very expansive and allows many different extensions of functionality, depending on future research.

1.3 Chapter overview

The main results of this thesis are supported by a plethora of definitions and smaller results. We summarise the contents of each chapter for convenience.

In Chapter 2, we define three different formalisms for combinatorial surfaces: polygonal complexes, twisted polygonal complexes, and Dress surfaces. Each of these formalisms is useful in different circumstances, but the introduction of several formalisms for "the same concept" leads to duplication of work. To ameliorate this, we follow a categorical approach and define functors that convert between the formalisms. Then, we introduce the abstractions of *combinatorial complex* (Definition 2.8.1) and *combinatorial properties* (Definition 2.8.2) to de–emphasise the concrete formalism.

Chapter 3 is concerned with the graph–theoretical aspects of combinatorial surfaces. It defines *vertex–edge–graph* and *face–edge–graph*, whose colourings correspond to certain surface colourings. Via this correspondence, the literature on graph colourings is made available for colourings of combinatorial surfaces.

To deal with boundary graphs of combinatorial surfaces, we develop a formalism for cyclic graphs in Section 3.4 and generalise the concept of *interval* to the cyclic case.

In Chapter 4, *degree* and *defect* are introduced. We also develop the formalism of *extended combinatorial surfaces*, allowing us to treat every surface with boundary as subsurface of a larger combinatorial surface. In Section 4.3 we present the standard correspondence between regular surfaces and certain subgroups of triangle groups.

Chapter 5 explores the topological aspects of combinatorial surfaces. It starts by formally constructing the topological realisation of a combinatorial surface in our formalism. Then, Section 5.2 defines connectivity and strong connectivity. In particular, it links combinatorial connectivity to connectivity of the topological realisation.

Section 5.3 defines orientation for combinatorial surfaces. In addition to this purely topological concept, we develop *dual orientation*, which is a mostly combinatorial property that is intimately linked with orientation.

Chapter 6 showcases several surface modifications and proves their formal correctness.

Chapter 7 introduces the hexagonal lattice, well–known in the literature. In Section 7.2, we develop a precise formalism for closed paths in the lattice. This allows us to abstractly construct closed paths with certain properties.

1.4 Summary

This thesis explores four different aspects of combinatorial surfaces: nets, modifications, classification, and implementation. In each of these topics this thesis contributes new results and new software.

In addition to these, another important aspect of this thesis is the groundwork it has laid. Specifically, it introduced several new methods of proof (like the construction of the infinite regular extension in Section 8.3) and there are several new perspectives offered (like the differential–geometric perspective in Section 7.3). We hope that this has paved the way for future research to explore these concepts.

1.5 Notation

In this section, we recall some basic concepts that are used throughout this thesis. A general feature of this thesis are the “well–defined” environments that follow some of the definitions. Similar to proof–environments, they contain a proof for the well–definedness of the definition.

We sometimes have to round up or round down.

Definition 1.5.1. *Let $x \in \mathbb{R}$. Then,*

$$\lceil x \rceil := \min\{k \in \mathbb{Z} \mid k \geq x\}$$
$$\lfloor x \rfloor := \max\{k \in \mathbb{Z} \mid k \leq x\}.$$

1.5.1 Set theory

We often refer to disjoint unions and power sets.

Definition 1.5.2. *Let A and B be two sets. The **disjoint union** $A \uplus B$ is the set*

$$\{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\}.$$

If the meaning is clear, we say $a \in A \uplus B$ to refer to $(a, 1) \in A \uplus B$ (similar for $b \in B$).

Next, we define the power set of a set, together with the set of all subsets with the same cardinality.

Definition 1.5.3. *Let M be a set. Its **power set** $\mathrm{Pot}(M)$ is the set of all subsets of M. For $k \in \mathbb{Z}$, we define*

$$\mathrm{Pot}_k(M) := \{x \in \mathrm{Pot}(M) \mid |x| = k\}.$$

Another often used concept is the *difference* of sets.

Definition 1.5.4. *Let A and B be sets. Their **difference** $A \backslash B$ is the set*

$$\{x \in A \mid x \notin B\}.$$

We sometimes employ equivalence relations and equivalence classes, so we recall their definitions here, together with the notation we are using.

Definition 1.5.5. *Let M be a set. A relation $\sim \subseteq M \times M$ is an **equivalence relation** if it is*

1. ***reflexive**, i. e. $m \sim m$ for all $m \in M$,*

2. ***symmetric****, i. e. $m \sim n$ implies $n \sim m$ for all $m, n \in M$, and*

3. ***transitive****, i. e. $m \sim n$ and $n \sim p$ imply $m \sim p$ for all $m, n, p \in M$.*

If $\sim$ is an equivalence relation, the set $\{n \in M \mid n \sim m\}$ is the ***equivalence class*** *of an $m \in M$, usually denoted $[m]$ or $[m]_\sim$.*

1.5.2 Group theory

At several points in this thesis, we employ group theory. We assume the reader is familiar with the elementary definitions of group, subgroup, normal subgroup, group actions and presentations.

Definition 1.5.6. *Let M be a set. The* ***symmetric group on*** *M, denoted $\mathrm{Sym}(M)$, is the group of all bijective maps $M \to M$.*

We often write the elements of the symmetric group in cycle notation. For $M = \{1, 2, 3, 4, 5, 6\}$, the element $(1, 2, 3)(5, 6)$ denotes the bijection

$$1 \mapsto 2, \qquad 2 \mapsto 3, \qquad 3 \mapsto 1, \qquad 4 \mapsto 4, \qquad 5 \mapsto 6, \qquad 6 \mapsto 5.$$

Definition 1.5.7. *Let $n \geq 1$. The* ***dihedral group*** *is the subgroup of $\mathrm{Sym}(\{1, \ldots, 2n\})$, generated by these bijections:*

$$k \mapsto \begin{cases} k+1 & k < 2n \\ 1 & k = 2n \end{cases} \qquad\qquad k \mapsto 2n + 1 - k$$

Concerning group actions, we generally act from the left and denote the action of the group element g on the element x by $g.x$.

We fix the notation for closure and normal closure.

Definition 1.5.8. *Let G be a group and $g_1, \ldots, g_n \in G$. The subgroup generated by $g_1, \ldots, g_n$ is denoted $\langle g_1, \ldots, g_n \rangle$. The normal subgroup generated by $g_1, \ldots, g_n$ is denoted $\langle\langle g_1, \ldots, g_n \rangle\rangle$ and called the* ***normal closure***

We write the presentation of a group in a similar fashion. For example, the dihedral group of order $2n$ has the presentation $\langle x, y \mid x^2, y^2, (xy)^n \rangle$.

Furthermore, we use the normaliser and the semi–direct product.

Definition 1.5.9. *Let G be a group and $U \leq G$ a subgroup. The* ***normaliser*** *of U in G is the subgroup*

$$N_G(U) := \{g \in G \mid gUg^{-1} = U\}.$$

We denote the semi–direct product by the symbol $\ltimes$.

Definition 1.5.10. *Let N and H be groups, with a group homomorphism $\mu : H \to \mathrm{Aut}(N)$. The* ***semi–direct product*** *$H \ltimes N$ is the group with base set $H \times N$ and product*

$$(h_1, n_1) \cdot (h_2, n_2) := (h_1 h_2, \mu(h_1)(n_1) n_2).$$

2 Definitions of combinatorial surfaces

2.1 Overview

The main focus of this thesis is combinatorial surfaces. One class of combinatorial surfaces arises when considering the triangulation of a surface only combinatorially (that is, we only care about the relation of its vertices, edges, and faces), this object is a combinatorial surface. In such a triangulation, all faces are triangles, but we will often use general polygons.

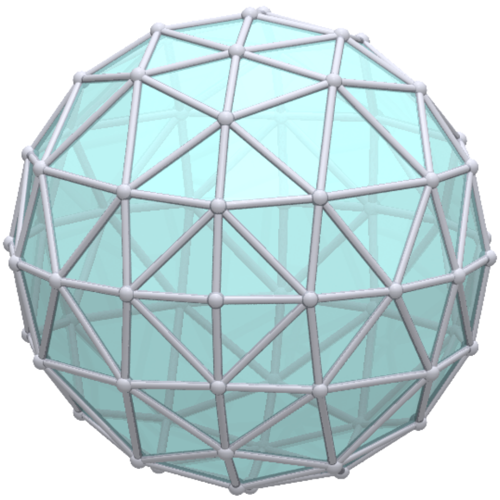
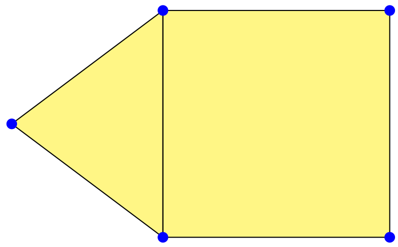
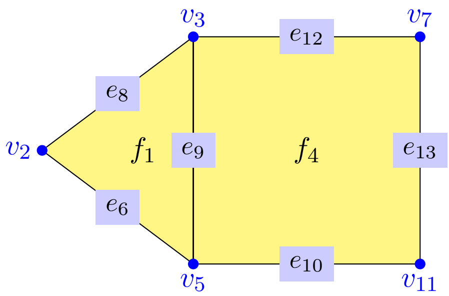


Figure 2.1: Triangulation of a sphere ([53])

In the field of combinatorial surfaces, it is quite easy to roughly point to the objects that are studied ("just look at a picture!"). But to actually work with them, we need a formal definition of these objects. This turns out to be harder than one initially thinks. For example, consider this picture:



We would like to have a formalisation of the object in the picture. To distinguish between similar objects, we label them.



A simple formalisation consists of the following data:

- A set V of vertices, a set E of edges, and a set F of faces.

  In our example, we have the vertices $V = \{v_2, v_3, v_5, v_7, v_{11}\}$, the edges $E = \{e_6, e_8, e_9, e_{10}, e_{12}, e_{13}\}$, and the faces $F = \{f_1, f_4\}$.

- A transitive incidence relation $\prec$ between vertices, edges, and faces.

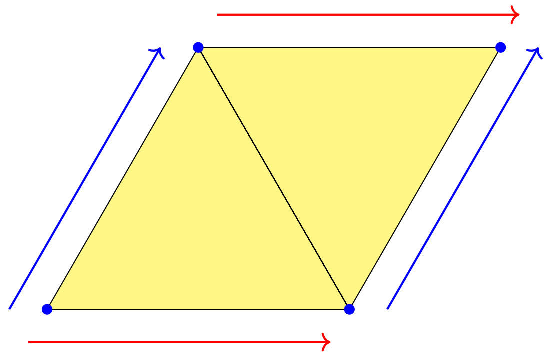
  In our example, we have for example $v_3 \prec e_9 \prec f_4$, but $v_2 \not\prec e_9$.

- A non–degeneracy condition that forbids a vertex to be incident to a face twice. In particular, a triangular face has exactly three vertices and three edges.

- Several properties that have to be fulfilled to make the incidence relation correspond to our intuition about "surfaces".

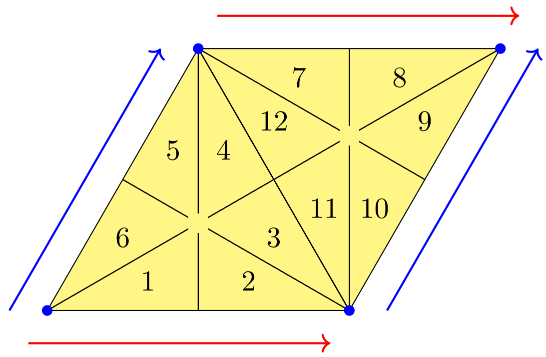
This leads to the concept *polygonal surface* (Subsection 2.5.2). It is a pretty simple model and allows to formulate many statements.

Unfortunately, it fails for a few applications. Consider the following object, where the edges are identified according to the coloured arrows.



In this object, there is only one vertex. This conflicts with the requirement that a triangular face has three distinct vertices. Thus, this object cannot be modelled by a polygonal surface. The solution is to consider *chambers*. For polygonal surfaces, they could be defined as triples consisting of a vertex, an edge, and a face, all of which are incident to each other.

In our example, we can start with the set of chambers:



To be able to reconstruct the torus, we need to know which chambers are adjacent. This can be done by involutions. If we define the surface completely by chambers and their adjacencies, we obtain *Dress–surfaces*.

Since we sometimes want to consider more general objects than surfaces, we can also combine the formalisms of polygonal surfaces and Dress–surfaces. This yields *twisted polygonal surfaces*.

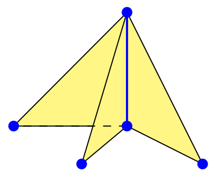
In the literature, there are many more formalisations of these objects, ranging from triangulations ([49]) to combinatorial maps ([41] and [14]), including embeddings of (three–regular) graphs ([17] and [18]). All of them point to similar phenomena, so it is only expected that they agree on many of their implications.

Instead of introducing all of these theories and showing the correspondences, we focus on the three formalisms of polygonal surfaces, twisted polygonal surfaces, and Dress–surfaces. We show how these formalisms can be related to each other and which results can be transferred.

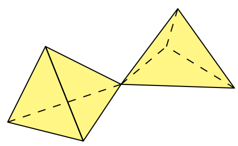
2.1.1 Generalisations of surfaces

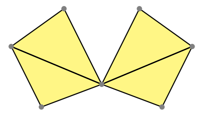
It is sometimes convenient to study objects that are a bit more general than combinatorial surfaces, but still build from polygons. We work with two essential generalisations: ramified edges and ramified vertices.

A ramified edge is an edge that is incident to at least three faces.



The definition of a ramified vertex is more involved. The detailed formalism will be postponed until Subsection 2.5.2 (for polygonal surfaces) and Subsection 2.4.1 (for twisted polygonal surfaces). To give a rough idea how a ramified vertex looks like, consider these illustrations:





Polygonal complexes are the more general object compared to polygonal surfaces. Likewise, *twisted polygonal complexes* generalise twisted polygonal surfaces. The concept of Dress–surfaces cannot be generalised in this fashion since its definitions of vertices and edges preclude ramifications of this kind.

2.1.2 Categorical overview

At this point, we already mentioned three different formalisms, together with two different specifications for some of them. Since this multitude of approaches can quickly become overwhelming, we employ the language of category theory for an easier overview.

$$\begin{tikzcd}
\mathbf{DressSurf} \arrow[d, hook] & & \\
\mathbf{TwistPolySurf} \arrow[rr, hook] & & \mathbf{TwistPolyComp} \\
\mathbf{PolySurf} \arrow[u, hook] \arrow[rr, hook] & & \mathbf{PolyComp} \arrow[u, hook] \\
& \mathbf{TriComp} \arrow[ur, hook] & \\
& \mathbf{SimpComp}^2 \arrow[u, hook] &
\end{tikzcd}$$

In this overview, the new categories **TriComp** and **SimpComp**^{2} appear. **TriComp** consists of all polygonal complexes with only triangles as faces. **SimpComp**^{2} consists of simplicial complexes. As the image suggests, some triangular complexes can be described as simplicial complexes. We call those *vertex–faithful* (compare Subsection 2.7.1).

2.2 Background: Category theory

In this section, we provide the essential definitions for our use of categorical language. They can be found in almost all introductory textbooks, like [1] and [50]. In our presentation, we mostly follow [1].

Definition 2.2.1. *A* ***category*** *is a quadruple* $\mathcal{A} = (\mathcal{O}, \hom, id, \circ)$ *consisting of:*

1. *A class* $\mathcal{O}$*, whose elements are called* $\mathcal{A}$*–**objects**. It is usually referred to as* $ob(\mathcal{A})$*.*
2. *For each pair* (A, B) *of* $\mathcal{A}$*–objects, a set* $\hom(A, B)$*, whose elements are called* $\mathcal{A}$*–**morphisms from*** A ***to*** B*. We write* $f : A \to B$ *instead of* $f \in \hom(A, B)$*.*
3. *For each* $\mathcal{A}$*–object* A*, a morphism* $id : A \to A$*, called the* $\mathcal{A}$*–**identity** on* A*.*
4. *A* ***composition law*** *associating with each* $\mathcal{A}$*–morphism* $f : A \to B$ *and each* $\mathcal{A}$*–morphism* $g : B \to C$ *an* $\mathcal{A}$*–morphism* $g \circ f : A \to C$*.*

They have to satisfy the following conditions:

1. *Composition is associative.*
2. $\mathcal{A}$*–identities act as identities with respect to composition.*
3. *The sets* $\hom(A, B)$ *are pairwise disjoint.*

Definition 2.2.2. *Let* $\mathcal{A}$ *be a category. A morphism* $f : A \to B$ *is called* ***isomorphism*** *if there is a morphism* $g : B \to A$ *such that* $f \circ g = id_B$ *and* $g \circ f = id_A$*.*

Definition 2.2.3. *The category* $\mathcal{A}$ *is called* ***subcategory*** *of the category* $\mathcal{B}$ *if the following conditions are satisfied:*

1. $ob(\mathcal{A}) \subseteq ob(\mathcal{B})$*.*
2. *For each* $A, A' \in ob(\mathcal{A})$*, we have* $\hom_{\mathcal{A}}(A, A') \subseteq \hom_{\mathcal{B}}(A, A')$*.*
3. *For each* $\mathcal{A}$*–object* A*, the* $\mathcal{B}$*–identity on* A *is the* $\mathcal{A}$*–identity on* A*.*
4. *The composition law in* $\mathcal{A}$ *is the restriction of the composition law in* $\mathcal{B}$ *to the morphisms of* $\mathcal{A}$*.*

If, additionally, $\hom_{\mathcal{A}}(A, A') = \hom_{\mathcal{B}}(A, A')$ *holds for all* $A, A' \in ob(\mathcal{A})$*, we call* $\mathcal{A}$ *a* ***full subcategory*** *of* $\mathcal{B}$*.*

We employ two main constructions to construct subcategories: Restriction of the objects and restrictions of the morphisms.

Remark 2.2.4. *Let $\mathcal{A}$ be a category.*

1. *Let $S \subseteq ob(\mathcal{A})$. Then, $(S, \hom, id, \circ)$ is a full subcategory of $\mathcal{A}$.*
2. *For all pairs $A, A' \in \mathcal{A}$, let $H_{A,A'} \subseteq \hom(A, A')$ such that*
   - *$\bigcup_{A,A'} H_{A,A'}$ is closed under composition*
   - *The identity on A is contained in $H_{A,A}$.*

   Then, $(ob(\mathcal{A}), H_S, id, \circ)$ is a subcategory of $\mathcal{A}$, with $H_S(A, A') := H_{A,A'}$.

To connect different categories, we need the concept of *functor*.

Definition 2.2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **functor from $\mathcal{A}$ to $\mathcal{B}$** is a function that assigns each object $A \in ob(\mathcal{A})$ an object $F(A) \in ob(\mathcal{B})$, and each $\mathcal{A}$–morphism $f : A \to A'$ a $\mathcal{B}$–morphism $F(f) : F(A) \to F(A')$, such that*

1. *$F(f \circ g) = F(f) \circ F(g)$, whenever $f \circ g$ is defined, and*
2. *$F(id_A) = id_{F(A)}$ for all $A \in ob(\mathcal{A})$.*

Functors can also be restricted.

Remark 2.2.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between the categories $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{A}'$ and $\mathcal{B}'$ be subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. If*

1. *$F(A') \in ob(\mathcal{B}')$ for all $A' \in ob(\mathcal{A}')$ and*
2. *$F(f)$ is a $\mathcal{B}'$–morphism for each $\mathcal{A}'$–morphism f,*

F restricts to a functor from $\mathcal{A}'$ to $\mathcal{B}'$.

Functors can have several nice properties.

Definition 2.2.7. *Let $F : \mathcal{A} \to \mathcal{B}$ a functor.*

1. *F is called **faithful** if the induced maps on morphisms*
$$\hom_{\mathcal{A}}(A, A') \to \hom_{\mathcal{B}}(F(A), F(A')) \qquad f \mapsto F(f)$$
   are injective.
2. *F is called **full** if the induced maps on morphisms*
$$\hom_{\mathcal{A}}(A, A') \to \hom_{\mathcal{B}}(F(A), F(A')) \qquad f \mapsto F(f)$$
   are surjective.
3. *F is called **essentially surjective** if for any object $B \in ob(\mathcal{B})$, there exists an object $A \in ob(\mathcal{A})$ such that $F(A)$ is isomorphic to B.*
4. *F is called **equivalence of categories** if it is faithful, full, and essentially surjective.*

We note that *essentially surjective* functors are called *isomorphism–dense* in [1, Definition 3.33].

2.3 Category $\mathrm{SimpComp}^2$ of simplicial complexes

In the literature, there is no consensus about the definition of simplicial complexes. The term can refer to

1. A strictly combinatorial object ([62, Section 3.1], [8, Chapter 12]), which is sometimes denoted as *abstract simplicial complex* ([47, Section 2.1]).
2. A collection of simplices (as subsets of $\mathbb{R}^n$ or a more general Euclidean space) that satisfies certain intersection criteria ([59, Chapter 7], [61, Chapter 4], [63, Section 0.2], [60, Section 2.10]). In this case, the combinatorial object is sometimes called a *schema*.

We will mostly concern ourselves with the first, strictly combinatorial definition.

Definition 2.3.1. *Let V be a set and $\Delta \subseteq \mathrm{Pot}(V)\backslash\{\emptyset\}$. Then, (V, Δ) is a **simplicial complex** if every non–empty subset of a set in Δ is also contained in Δ.*
*It is called **finite** if V is finite and every element of Δ is finite.*

There are several different maps between simplicial complexes.

Definition 2.3.2. *Let (V_1, Δ_1) and (V_2, Δ_2) be two simplicial complexes. We call a map $\mu : V_1 \to V_2$ a*

- ***simplicial morphism** if $x \in \Delta_1$ implies $\{\mu(v) \mid v \in x\} \in \Delta_2$.*
- ***simplicial shadow morphism** if $x \in \mathrm{Pot}(V_1)\backslash\Delta_1$ implies*
$$\{\mu(v) \mid v \in x\} \in \mathrm{Pot}(V_2)\backslash\Delta_2.$$
- ***simplicial twilight morphism** if μ is both a simplicial morphism and a simplicial shadow morphism.*

*A simplicial morphism μ is called **dimension–preserving** if for every $x \in \Delta_1$ we have $|x| = |\{\mu(v) \mid v \in x\}|$.*

We can characterise simplicial shadow morphisms in a different way.

Remark 2.3.3. *Let (V_1, Δ_1) and (V_2, Δ_2) be simplicial complexes. A map $\mu : V_1 \to V_2$ is a simplicial shadow morphism if and only if for every $x \in \mathrm{Pot}(V_1)$ the implication*
$$\{\mu(y) \mid y \in x\} \in \Delta_2 \quad \Rightarrow \quad x \in \Delta_1$$
holds.

Proof. Given $x \in \mathrm{Pot}(V_1)$ and $\mu(x) := \{\mu(y) \mid y \in x\}$, the following statements are logically equivalent:
$$\begin{aligned} & x \notin \Delta_1 \Rightarrow \mu(x) \notin \Delta_2 \\ \Leftrightarrow\quad & \neg(x \notin \Delta_1) \vee \mu(x) \notin \Delta_2 \\ \Leftrightarrow\quad & \neg(\mu(x) \in \Delta_2) \vee x \in \Delta_1 \\ \Leftrightarrow\quad & \mu(x) \in \Delta_2 \Rightarrow x \in \Delta_1 \end{aligned}$$
□

Remark 2.3.4. *A dimension–preserving simplicial twilight morphism between finite simplicial complexes is injective.*

Proof. Let $\mu : (V_1, \Delta_1) \to (V_2, \Delta_2)$ be the dimension–preserving simplicial twilight morphism and consider $x, y \in \Delta_1$ with $\{\mu(u) \mid u \in x\} = \{\mu(v) \mid v \in y\}$. In particular, their union $\{\mu(u) \mid u \in x\} \cup \{\mu(v) \mid v \in y\}$ also lies in Δ_2. Since μ is a simplicial shadow morphism, $x \cup y \in \Delta_1$. But, if $x \neq y$, then $|x \cup y| > |x| = |\{\mu(u) \mid u \in x\}|$, which implies that $x \cup y$ is mapped to a set of smaller cardinality. □

Definition 2.3.5. *Let (V, Δ_V) be a simplicial complex and $W \subseteq V$. The simplicial complex (W, Δ_W) with $\Delta_W := \{x \in \Delta_V \mid x \subseteq W\}$ is the **induced subcomplex**. The map $\iota_W : (W, \Delta_W) \to (V, \Delta_V), v \mapsto v$ is the **natural inclusion**.*

Remark 2.3.6. *Let (V, Δ_V) be a simplicial complex and $W \subseteq V$. The natural inclusion $\iota_W : (W, \Delta_W) \to (V, \Delta_V), v \mapsto v$ is a simplicial twilight morphism.*

Proof. Since $\Delta_W \subseteq \Delta_V$, the map ι is a simplicial morphism. Since $\Delta_W = \Delta_V \cap \mathrm{Pot}(W)$, we know $x \in \mathrm{Pot}(W) \backslash \Delta_W$ implies $x \notin \Delta_V$. □

If some vertices do not lie in the image of a morphism, these images can be ignored.

Lemma 2.3.7. *Let $\mu : (V_1, \Delta_1) \to (V_2, \Delta_2)$ be a simplicial (shadow/twilight) morphism. For each $W \subseteq V_2$ with $\mu(V_1) \subseteq W$ there is a unique simplicial (shadow/twilight) morphism $\rho : (V_1, \Delta_1) \to (W, \Delta_W)$ with $\iota \circ \rho = \mu$, where ι is the natural inclusion from Definition 2.3.5.*

Proof. We have to show that the map ρ exists and that it is unique. From $\iota \circ \rho = \mu$, we deduce that $v \in V_1$ has to be mapped to $\mu(v) \in W$. This defines ρ uniquely. Next, we have to show:

- If μ is a simplicial morphism, so is ρ.
- If μ is a simplicial shadow morphism, so is ρ.

If μ is a simplicial morphism, $x \in \Delta_1$ implies $\mu(x) \in \Delta_2 \cap \mathrm{Pot}(W) = \Delta_W$. If μ is a shadow morphism, $x \in \mathrm{Pot}(V_1) \backslash \Delta_1$ implies $\mu(x) \in \mathrm{Pot}(V_2) \backslash \Delta_2$. Since $\mu(x) \in \mathrm{Pot}(W)$, we have $\mu(x) \in \mathrm{Pot}(W) \cap (\mathrm{Pot}(V_2) \backslash \Delta_2) = \mathrm{Pot}(W) \backslash \Delta_W$. □

Simplicial complexes form a category, but we focus on a more restricted set.

Definition 2.3.8. *Let (V, Δ) be a simplicial complex.*

- *Its **dimension** is* $\max_{x \in \Delta} |x| - 1$.
- *It is **homogeneous** if every simplex is contained in a simplex of maximal size.*

Definition 2.3.9. $\mathbf{SimpComp}^2$ *refers to the category of homogeneous simplicial complexes with dimension 2, together with simplicial morphisms.*

Well–defined. We have to check the properties of Definition 2.2.1.

Clearly, the identity morphism is a simplicial morphism and the composition of simplicial morphisms (as maps) gives another simplicial morphism. The remaining properties are trivially satisfied. □

2.4 Category TwistPolyComp of twisted polygonal complexes

This section describes the most general surface structure considered in this thesis, the twisted polygonal complex. It consists of the following data:

- Sets of vertices V, edges E, faces F, and chambers C.
- Each chamber consists of a vertex, an edge, and a face, which is encoded by a map $\lambda : C \to V \times E \times F$. Geometrically, the chambers correspond to the barycentric subdivision of the combinatorial surface.
- The involution $\sigma_0 : C \to C$ encodes the adjacency of two chambers that lie in the same polygon and differ only in their vertex.
- The involution $\sigma_1 : C \to C$ encodes the adjacency of two chambers that lie in the same polygon and differ only in their edge.
- To encode the adjacency of chambers only differing in their face, we cannot use an involution since there might be more than two chambers satisfying these criteria (if an edge is ramified). Thus, we employ an equivalence relation $\sim$ for this task.

We note that the images of σ_0 and σ_1 are not defined uniquely by our previous description. For example, consider the two–torus in Figure 2.2.

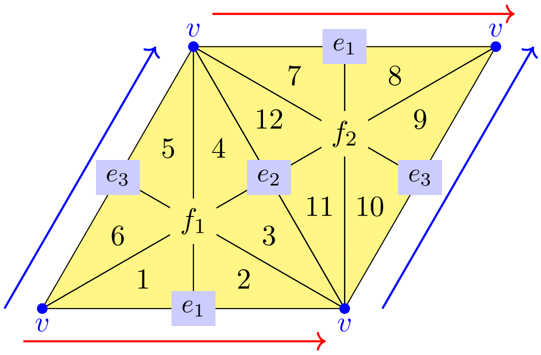


Figure 2.2: The two–torus as twisted polygonal complex

In this illustration, there is only one vertex v, the edges are $\{e_1, e_2, e_3\}$, and the faces are $\{f_1, f_2\}$. The chambers are represented by the integers from 1 to 12. Since σ_0 and σ_1 encode adjacency, we can read them off from the picture:

$$\sigma_0 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$$
$$\sigma_1 = (1,6)(2,3)(4,5)(7,12)(8,9)(10,11)$$

But the chambers 1 and 2 have the same vertex, the same edge, and the same face. Thus, the involution σ_0 carries more information than encoded in the incidence.

In particular, it is possible that several vertices within a polygon coincide (for the two–torus, all vertices coincide).

Definition 2.4.1. *A* ***twisted polygonal complex*** *is an 8–tuple* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *such that:*

- V *is a set called* ***vertices****,* E *is a set called* ***edges****,* F *is a set called* ***faces****,* C *is a set called* ***chambers****.*
- *The map* $\lambda : C \to V \times E \times F$ *is called* ***flag map****. Its projections are:*

$$\begin{aligned} \lambda_0 &:= ((v, e, f) \mapsto v) \circ \lambda & \lambda_{01} &:= ((v, e, f) \mapsto (v, e)) \circ \lambda \\ \lambda_1 &:= ((v, e, f) \mapsto e) \circ \lambda & \lambda_{02} &:= ((v, e, f) \mapsto (v, f)) \circ \lambda \\ \lambda_2 &:= ((v, e, f) \mapsto f) \circ \lambda & \lambda_{12} &:= ((v, e, f) \mapsto (e, f)) \circ \lambda. \end{aligned}$$

- $\sigma_0 : C \to C$ *is an involution without fixed points, such that* $\lambda_{12} = \lambda_{12} \circ \sigma_0$*.*
- $\sigma_1 : C \to C$ *is an involution without fixed points, such that* $\lambda_{02} = \lambda_{02} \circ \sigma_1$*.*
- $\sim$ *is an equivalence relation on* C*, such that* $c_1 \sim c_2$ *implies both* $\sigma_0(c_1) \sim \sigma_0(c_2)$ *and* $\lambda_{01}(c_1) = \lambda_{01}(c_2)$*. The equivalence class of* $c \in C$ *is denoted by* $[c]_\sim$*.*
- *Two chambers* $c_1, c_2 \in C$ *with* $\lambda_1(c_1) = \lambda_1(c_2)$ *satisfy* $c_1 \sim c_2$ *or* $c_1 \sim \sigma_0(c_2)$*.*
- *Two chambers* $c_1, c_2 \in C$ *with* $\lambda_2(c_1) = \lambda_2(c_2)$ *satisfy* $c_1 \in \langle \sigma_0, \sigma_1 \rangle .c_2$*.*

Example 2.4.2. *The two–torus* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *with*

$$V = \{v\}, \qquad E = \{e_1, e_2, e_3\}, \qquad F = \{f_1, f_2\}, \qquad C = \{c_1, \dots, c_{12}\},$$

and

$$\lambda : C \to V \times E \times F, \qquad c_k \mapsto \begin{cases} (v, e_1, f_1) & k \in \{1, 2\} \\ (v, e_2, f_1) & k \in \{3, 4\} \\ (v, e_3, f_1) & k \in \{5, 6\} \\ (v, e_1, f_2) & k \in \{7, 8\} \\ (v, e_3, f_2) & k \in \{9, 10\} \\ (v, e_2, f_2) & k \in \{11, 12\} \end{cases}$$

$$\begin{aligned} \sigma_0 &= (c_1, c_2)(c_3, c_4)(c_5, c_6)(c_7, c_8)(c_9, c_{10})(c_{11}, c_{12}) \\ \sigma_1 &= (c_1, c_6)(c_2, c_3)(c_4, c_5)(c_7, c_{12})(c_8, c_9)(c_{10}, c_{11}) \\ \sim &: \{c_1, c_7\}, \{c_2, c_8\}, \{c_3, c_{11}\}, \{c_4, c_{12}\}, \{c_5, c_9\}, \{c_6, c_{10}\}, \end{aligned}$$

is a twisted polygonal complex that is illustrated in Figure 2.2.

Although incidence between vertices, edges, and faces is not part of Definition 2.4.1, it can be reconstructed from the map λ.

Definition 2.4.3. *Let* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *be a twisted polygonal complex. The relation* ***incidence*** $\prec \subseteq (V \times E) \uplus (V \times F) \uplus (E \times F)$ *is defined as follows:*

- *For $v \in V$ and $e \in E$, we have $v \prec e$ if $\lambda_{01}(c) = (v, e)$ for a chamber $c \in C$.*
- *For $v \in V$ and $f \in F$, we have $v \prec f$ if $\lambda_{02}(c) = (v, f)$ for a chamber $c \in C$.*
- *For $e \in E$ and $f \in F$, we have $e \prec f$ if $\lambda_{12}(c) = (e, f)$ for a chamber $c \in C$.*

Remark 2.4.4. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex. Incidence is a transitive relation.*

Proof. The vertex $v \in V$ is incident to the edge $e \in E$, if there is a chamber $c_1 \in C$ with $\lambda(c_1) = (v, e, \hat{f})$ for some face $\hat{f} \in F$.

The edge $e \in E$ is incident to the face $f \in F$, if there is a chamber $c_2 \in C$ with $\lambda(c_2) = (\hat{v}, e, f)$ for some vertex $\hat{v} \in V$.

Since $\lambda_1(c_1) = \lambda_1(c_2)$, either $c_1 \sim c_2$ or $c_1 \sim \sigma_0(c_2)$ holds. In the first case, we conclude from $(v, e) = \lambda_{01}(c_1) = \lambda_{01}(c_2) = (\hat{v}, e)$ that $v = \hat{v}$, which leads to $\lambda_{02}(c_2) = (v, f)$, proving $v \prec f$.

Otherwise, $(v, e) = \lambda_{01}(c_1) = \lambda_{01}(\sigma_0(c_2))$. Combining this with $(e, f) = \lambda_{12}(c_2) = (\lambda_{12} \circ \sigma_0)(c_2)$, we obtain $\lambda(\sigma_0(c_2)) = (v, e, f)$. □

Next, we define morphisms between twisted polygonal complexes. A morphism between two twisted polygonal complexes should consist of maps for vertices, edges, faces, and chambers that are compatible with λ, σ_0, σ_1, and $\sim$. However, we want to enforce an additional constraint: We would like to enforce that the number of vertices in a face does not change under the morphism, e. g. a hexagonal face should not be mapped to a triangular face.

Definition 2.4.5. *A* ***twisted polygonal morphism*** *between the twisted polygonal complexes $(V^1, E^1, F^1, C^1, \lambda^1, \sigma_0^1, \sigma_1^1, \sim^1)$ and $(V^2, E^2, F^2, C^2, \lambda^2, \sigma_0^2, \sigma_1^2, \sim^2)$ consists of maps*

$$\mu_V : V^1 \to V^2, \qquad \mu_E : E^1 \to E^2, \qquad \mu_F : F^1 \to F^2, \qquad \mu_C : C^1 \to C^2,$$

satisfying:

- ***λ–compatible:*** *If $\lambda^1(c) = (v, e, f)$, then $\lambda^2(\mu_C(c)) = (\mu_V(v), \mu_E(e), \mu_F(f))$.*
- ***σ_0–compatible:*** *$\mu_C \circ \sigma_0^1 = \sigma_0^2 \circ \mu_C$.*
- ***σ_1–compatible:*** *$\mu_C \circ \sigma_1^1 = \sigma_1^2 \circ \mu_C$.*
- ***$\sim$–compatible:*** *If $c_1 \sim^1 c_2$, then $\mu_C(c_1) \sim^2 \mu_C(c_2)$.*
- ***non–degenerate:*** *The restriction of μ_C to $\lambda_2^{-1}(f)$ is bijective for every $f \in F$.*

We can reformulate the non–degeneracy condition.

Lemma 2.4.6. *Let $(V^1, E^1, F^1, C^1, \lambda^1, \sigma_0^1, \sigma_1^1, \sim^1)$ and $(V^2, E^2, F^2, C^2, \lambda^2, \sigma_0^2, \sigma_1^2, \sim^2)$ be two twisted polygonal complexes. If all orbits of $\langle \sigma_0^1, \sigma_1^1 \rangle$ are finite, we can replace the non–degeneracy condition of Definition 2.4.5 by:*

- *For any $c \in C$, we have $|\langle \sigma_0^1, \sigma_1^1 \rangle .c| = |\langle \sigma_0^2, \sigma_1^2 \rangle .\mu_C(c)|$.*

Proof. For any chamber $c \in C$, we have $\langle \sigma_0, \sigma_1 \rangle .c = \lambda_2^{-1}(\lambda_2(c))$. Since μ_C is compatible with σ_0 and σ_1, it restricts to a map

$$\langle \sigma_0^1, \sigma_1^1 \rangle .c \to \langle \sigma_0^2, \sigma_1^2 \rangle .\mu_C(c).$$

We show that this map is surjective: Let $x \in \langle \sigma_0^2, \sigma_1^2 \rangle .\mu_C(c)$, then $x = w(\sigma_0^2, \sigma_1^2).\mu_C(c)$, where $w(a,b)$ is a word in the free group generated by a and b. But then,

$$\mu_C(w(\sigma_0^1, \sigma_1^1).c) = w(\sigma_0^2, \sigma_1^2).\mu_C(c) = x.$$

Since the cardinality of both orbits is identical, this proves that μ_C induces a bijection between them. □

Definition 2.4.7. TwistPolyComp *is the category formed from twisted polygonal complexes and twisted morphisms.*

2.4.1 Strong paths and twisted polygonal surfaces

In Section 2.4, we formalised twisted polygonal complexes. In this subsection, we restrict to those complexes that correspond to combinatorial surfaces. Like described in Subsection 2.1.1, we have to avoid both edge ramifications and vertex ramifications.

It is relatively easy to define edge ramifications by counting how many faces are incident to one edge. Unfortunately, we cannot rely on the incidence relation $\prec$, as the following example shows:

Example 2.4.8. *The one–cone $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ with*

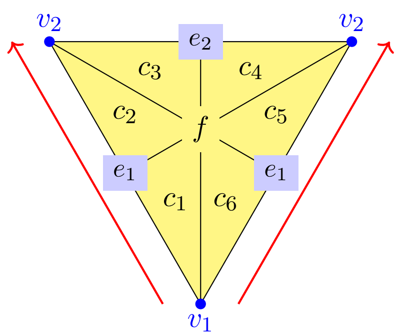
$$V = \{v_1, v_2\}, \qquad E = \{e_1, e_2\}, \qquad F = \{f\}, \qquad C = \{c_1, \dots, c_6\},$$

and

$$\lambda : C \to V \times E \times F, \qquad c_k \mapsto \begin{cases} (v_1, e_1, f) & k \in \{1,6\} \\ (v_2, e_1, f) & k \in \{2,5\} \\ (v_2, e_2, f) & k \in \{3,4\} \end{cases}$$

$$\sigma_0 = (c_1, c_2)(c_3, c_4)(c_5, c_6), \quad \sigma_1 = (c_1, c_6)(c_2, c_3)(c_4, c_5), \quad \sim : \{c_1, c_6\}, \{c_2, c_5\}, \{c_3\}, \{c_4\}$$

illustrated by



is a twisted polygonal complex.

There is a unique face, so every edge is incident to it. However, the sets $\lambda_1^{-1}(e_1)$ and $\lambda_1^{-1}(e_2)$ have different cardinality, corresponding to the "intuitive" view in which e_1 "looks like" an inner edge.

Therefore, we rely on λ_1^{-1} to distinguish between different types of edges.

Remark 2.4.9. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex.*

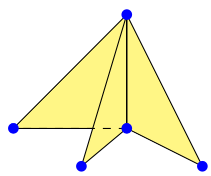
For each $e \in E$, the set $\lambda_1^{-1}(e)$ has an even number of elements.

Proof. Since σ_0 restricts to an involution without fixed points on $\lambda_1^{-1}(e) \to \lambda_1^{-1}(e)$, the claim follows. $\square$

Definition 2.4.10. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex.*

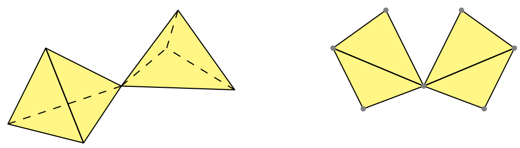
- *$e \in E$ is a* ***boundary edge*** *if $|\lambda_1^{-1}(e)| = 2$.*
- *$e \in E$ is an* ***inner edge*** *if $|\lambda_1^{-1}(e)| = 4$.*
- *$e \in E$ is a* ***ramified edge*** *if $|\lambda_1^{-1}(e)| > 4$.*

In Example 2.4.8, the edge e_1 is an inner edge, and the edge e_2 is a boundary edge. The central edge in



is a ramified edge.

The definition of vertex ramifications is more complicated. We want to avoid the situations depicted in these pictures:



In order to do so, we need to be able to talk about "the surface around a vertex". Since this is a local consideration, it should be sufficient to only consider chambers in which the vertex is contained. For a formal definition, we need the notion of *strong paths*. These are sequences of adjacent chambers.

Definition 2.4.11. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex and $c_1 \neq c_2 \in C$. Then,*

- *c_1 and c_2 are* ***0–adjacent*** *if $c_2 = \sigma_0(c_1)$.*

- *c_1 and c_2 are **1–adjacent** if $c_2 = \sigma_1(c_1)$.*

- *c_1 and c_2 are **2–adjacent** if $c_1 \sim c_2$.*

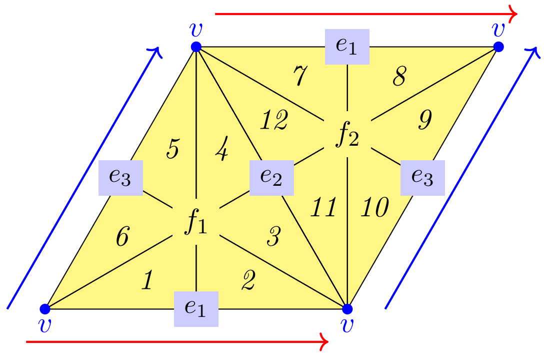
*c_1 and c_2 are **adjacent** if they are k–adjacent for at least one $k \in \{0, 1, 2\}$.*

Definition 2.4.12. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex.*

*A **strong path** is a sequence $(c_1, c_2, \ldots, c_n) \in C^n$ such that c_i and c_{i+1} are adjacent for all $1 \leq i < n$. A strong path is called **closed** if c_1 and c_n are adjacent. A strong path is called **non–repeating** if $c_i \neq c_j$ for $1 \leq i < j \leq n$.*

Our notion of a *closed* path differs from the usage in the graph theoretical literature. The main reason for this is that we want to build a correspondence to a different sort of paths later on (Lemma 2.5.22).

Example 2.4.13. *Consider the two–torus from Example 2.4.2:*



The sequence $(c_2, c_3, c_4, c_{12}, c_7)$ is a non–repeating strong path.

The sequence $(c_1, c_6, c_{10}, c_{11}, c_3, c_2)$ is a non–repeating closed strong path.

The sequence $(c_5, c_6, c_1, c_2, c_8, c_9, c_{10}, c_6, c_1)$ is a strong path that is neither closed nor non–repeating.

We can combine strong paths.

Definition 2.4.14. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex. Let $p = (c_1, \ldots, c_n)$ and $q = (d_1, \ldots, d_m)$ be two strong paths such that c_n and d_1 are adjacent. The **path–sum** $p + q$ is defined as the strong path $(c_1, \ldots, , c_n, d_1, \ldots, d_m)$.*

There are some special paths which we want to draw attention to. The first kind of paths stays within a single face.

Definition 2.4.15. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex.*

*A **strong polygon path** is a closed, non–repeating strong path $(c_1, c_2, \ldots, c_{2n}, c_{2n+1})$ such that c_k and c_{k+1} are 0–adjacent if k is even and 1–adjacent if k is odd.*

The strong polygon paths correspond to our native intuition about polygons.

Remark 2.4.16. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex.*

Any chamber $c \in C$ lies in exactly one strong polygon path (up to cyclic permutation and reflection of the entries).

Proof. Since σ_0 and σ_1 are involutions without fixed points, each chamber is 0–adjacent to exactly one other chamber (the same holds for 1–adjacency). Therefore, c lies in the unique strong polygon path $(c, \sigma_1(c), \sigma_0\sigma_1(c), \dots)$. □

Example 2.4.17. *In the two–torus from Example 2.4.2, the unique strong polygon paths are $(c_1, c_2, c_3, c_4, c_5, c_6)$ and $(c_7, c_8, c_9, c_{10}, c_{11}, c_{12})$.*

To define vertex ramifications, we are interested in strong paths that "surround" a single vertex, i. e. all of its chambers lie in $\lambda_0^{-1}(v)$ for a vertex v.

Definition 2.4.18. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex.*

*A **strong umbrella path** is a non–repeating strong path $(c_1, c_2, \dots, c_{2n})$ such that c_i and c_{i+1} are 2–adjacent if i is even and 1–adjacent if i is odd. If $\lambda_0(c_1) = v$, we sometimes denote this path as a **strong umbrella path around** v.*

*A strong umbrella path p around v is **maximal** if there is no strong umbrella path q around v such that $p + q$ or $q + p$ is a strong umbrella path.*

Remark 2.4.19. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex with a strong umbrella path $(c_1, c_2, \dots, c_{2n})$. Then, $\lambda_0(c_1) = \lambda_0(c_i)$ for all $1 \leq i \leq n$.*

If there are no ramified edges, we can characterise the maximal strong umbrella paths completely.

Corollary 2.4.20. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex without ramified edges and $(c_1, \dots, c_{2n})$ a maximal strong umbrella path. Then, one of the following cases holds:*

- *c_1 and c_{2n} are only $\sim$–equivalent to themselves.*
- *c_1 and c_{2n} are 2–adjacent.*

Proof. Assume we are not in the first case and consider the equivalence class $[c_{2n}]_\sim$. If this class is equal to $\{c_1, c_{2n}\}$, the path could be extended to a closed one, in contradiction to its maximality. Otherwise, there exists a $c \in C$ with $c \sim c_{2n}$ but $c \notin \{c_1, c_{2n}\}$. In particular, the strong path $(c_1, \dots, c_{2n}, c, \sigma_1(c))$ fulfils the adjacency condition for strong umbrella paths.

But the original path was maximal, so either c or $\sigma_1(c)$ is equal to a c_i with $1 \leq i \leq 2n$. We consider both cases in turn.

If $c = c_i$ for $1 < i < 2n$, the $\sim$–equivalence class of c contains at least three elements. This is only possible for a ramified edge.

On the contrary, assume $\sigma_1(c) = c_i$ for $1 \leq i \leq 2n$. This implies $c = \sigma_1(c_i)$. But $\sigma_1(c_i) \in \{c_1, \dots, c_{2n}\}$ by Definition 2.4.18. Thus, this reduces to the previous case. □

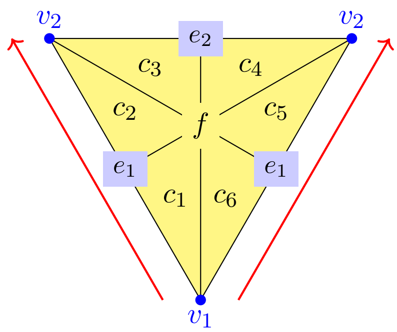
Lemma 2.4.21. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex without ramified edges. Then, every chamber $c \in C$ lies in exactly one maximal strong umbrella path (up to cyclic permutation and reflection).*

Proof. If there are no ramified edges, we can define an involution σ_2:

$$\sigma_2 : C \to C \qquad\qquad c \mapsto \begin{cases} c & [c]_\sim = \{c\} \\ \hat{c} & [c]_\sim = \{c, \hat{c}\} \neq \{c\} \end{cases}$$

Since every chamber is 1–adjacent to exactly one other chamber and is 2–adjacent to at most one other chamber, maximal strong umbrella paths correspond to the orbits of $\langle \sigma_1, \sigma_2 \rangle$. □

Example 2.4.22. *Consider the one–cone from Example 2.4.8:*



The unique maximal strong umbrella paths are (c_1, c_6, c_1) around v_1, and (c_3, c_2, c_5, c_4) around v_2.

In particular, if there are no ramified edges, the maximal strong umbrella paths partition the set $\lambda_0^{-1}(v)$ for each vertex $v \in V$. This allows us to define a vertex ramification: These are just the vertices where $\lambda_0^{-1}(v)$ is partitioned into more than one set.

Definition 2.4.23. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex.*

- *$v \in V$ is a **chaotic vertex**, if it is incident to a ramified edge.*
- *$v \in V$ is a **ramified vertex**, if it is not incident to a ramified edge and if there are at least two different maximal strong umbrella paths around v.*
- *$v \in V$ is a **boundary vertex**, if it is not incident to a ramified edge and there is a unique maximal strong umbrella path around v, which is non–closed.*
- *$v \in V$ is an **inner vertex**, if it is not incident to a ramified edge and there is a unique maximal strong umbrella path around v, which is closed.*

Finally, we can define twisted polygonal surfaces.

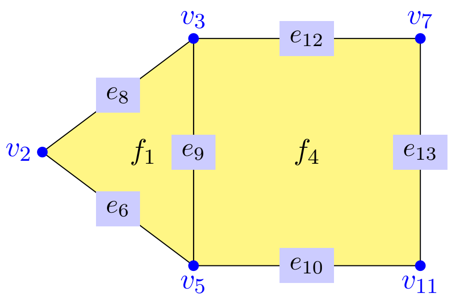
Definition 2.4.24. *A **twisted polygonal surface** is a twisted polygonal complex containing neither ramified vertices nor ramified edges.*

Combining Definition 2.4.7 and Remark 2.2.4, we obtain the category of twisted polygonal surfaces.

Definition 2.4.25. **TwistPolySurf** *is the category of twisted polygonal surfaces, together with twisted polygonal morphisms.*

2.5 Category PolyComp of polygonal complexes

In Section 2.4, we introduced the formalism of twisted polygonal complexes and surfaces. While it is very expressive, it is also quite unwieldy. If there is no need to study cases where several vertices or edges within a single face are identical, the formalism of polygonal complexes is usually more convenient. This is the case for the following combinatorial surface:



A polygonal complex is more flexible than a simplicial complex. For example, it is possible to have two edges with the same vertices. It is formalised as follows:

- Sets of vertices V, edges E, and faces F.
- A map $\eta : E \to \mathrm{Pot}_2(V)$ that associates each edge to its end–points.
- A map $\varphi : F \to \mathrm{Pot}(F)$ that associates each face to its incident edges.

To map a face to its incident vertices, we have to combine η and φ. Since we often use this combination, we give it a special name. As the definition is a combination of map composition (written as $\circ$) and set–theoretic union (written as $\cup$), we use the symbol ♉ for their combination. This symbol is called *Taurus* in astronomy.

Definition 2.5.1. *Let A, B, and C be sets. If $\alpha : A \to \mathrm{Pot}(B)$ and $\beta : B \to \mathrm{Pot}(C)$ are maps, then the **taurus composition** $\beta ♉ \alpha : A \to \mathrm{Pot}(C)$ is defined by $a \mapsto \bigcup_{b\in\alpha(a)} \beta(b)$.*

Definition 2.5.2. *Let V (**vertices**), E (**edges**), and F (**faces**) be sets with maps $\eta : E \to \mathrm{Pot}_2(V)$ and $\varphi : F \to \mathrm{Pot}(E)$. The quintuple (V, E, F, η, φ) is called **polygonal complex** if and only if*

1. Faces are polygons: *For every $f \in F$, there is a sequence $(v_1, e_1, v_2, e_2, \ldots, v_k, e_k)$ satisfying*
   - $k = |\varphi(f)| = |\eta ♉ \varphi(f)| \geq 3$.

- $\{e_1, e_2, \ldots, e_k\} = \varphi(f)$ *and* $\{v_1, \ldots, v_k\} = \eta \between \varphi(f)$.
- $\eta(e_i) = \{v_i, v_{i+1}\}$ *for all* $1 \leq i < k$ *and* $\eta(e_k) = \{v_1, v_k\}$.

2. Every vertex lies in an edge: *For every* $v \in V$*, there is an* $e \in E$ *with* $v \in \eta(e)$.

3. Every edge lies in a face: *For every* $e \in E$*, there is an* $f \in F$ *with* $e \in \varphi(f)$.

We define an ***incidence*** *relation* $\prec \subseteq (V \times E) \uplus (V \times F) \uplus (E \times F)$ *as follows:*

- *For* $v \in V$ *and* $e \in E$*, we have* $v \prec e$ *if and only if* $v \in \eta(e)$.
- *For* $e \in E$ *and* $f \in F$*, we have* $e \prec f$ *if and only if* $e \in \varphi(f)$.
- *For* $v \in V$ *and* $f \in F$*, we have* $v \prec f$ *if and only if there is an* $e \in E$ *with* $v \prec e$ *and* $e \prec f$.

In particular, the map η may be non–injective. This means that there are two edges whose incident vertices are identical. We usually work with finite polygonal complexes.

Example 2.5.3. (V, E, F, η, φ) *with*

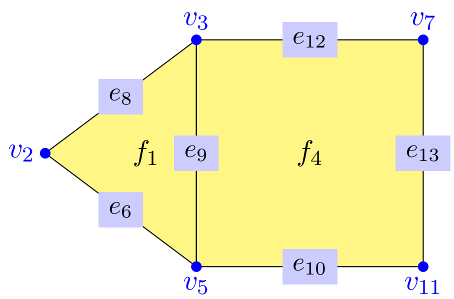
$$V = \{v_2, v_3, v_5, v_7, v_{11}\}, \qquad E = \{e_6, e_8, e_9, e_{10}, e_{12}, e_{13}\}, \qquad F = \{f_1, f_4\},$$

and

$$\eta : E \to \operatorname{Pot}_2(V) \qquad e \mapsto \begin{cases} \{v_2, v_5\} & e = e_6 \\ \{v_2, v_3\} & e = e_8 \\ \{v_3, v_5\} & e = e_9 \\ \{v_5, v_{11}\} & e = e_{10} \\ \{v_3, v_7\} & e = e_{12} \\ \{v_7, v_{11}\} & e = e_{13}, \end{cases}$$

$$\varphi : F \to \operatorname{Pot}(E) \qquad f \mapsto \begin{cases} \{e_6, e_8, e_9\} & f = f_1 \\ \{e_9, e_{10}, e_{11}, e_{13}\} & f = f_4, \end{cases}$$

illustrated by



is a polygonal complex. The sequence for f_1 *is* $(v_2, e_8, v_3, e_9, v_5, e_6)$*. The sequence for* f_4 *is* $(v_3, e_9, v_5, e_{10}, v_{11}, e_{13}, v_7, e_{12})$.

Sometimes we want to restrict attention to those polygonal complexes in which all polygons are triangles. We call those complexes *triangular complexes*.

Definition 2.5.4. *Let $P = (V, E, F, \eta, \varphi)$ be a polygonal complex. Define*

$$|\cdot| : F \to \mathbb{N} \qquad\qquad f \mapsto |\varphi(f)|.$$

*If $|f| = 3$ for all $f \in F$, we call P a **triangular complex**.*

Triangular complexes have a nice formal property: Any two–element–subset of vertices that are incident to a face, lies in the image of η. To see that this is not true in general, consider the polygonal complex of Example 2.5.3 and the set $\{v_3, v_{11}\}$.

Corollary 2.5.5. *Let $P = (V, E, F, \eta, \varphi)$ be a polygonal complex and $f \in F$ with $|f| = 3$. For any $x \in \mathrm{Pot}_2(V)$ with $x \subseteq (\eta \mho \varphi)(f)$, there is an edge $e \in E$ with $\eta(e) = x$.*

Proof. Since $|f| = 3$, there is a sequence $(v_1, e_1, v_2, e_2, v_3, e_3)$ with $\varphi(f) = \{e_1, e_2, e_3\}$ and $(\eta \mho \varphi)(f) = \{v_1, v_2, v_3\}$, that satisfies

$$\eta(e_1) = \{v_1, v_2\} \qquad \eta(e_2) = \{v_2, v_3\} \qquad \eta(e_3) = \{v_1, v_3\}.$$

Since these are all two–element–subsets of $\{v_1, v_2, v_3\}$, the claim follows. □

Next, we define morphisms between polygonal complexes. Since a polygonal complex consists of three sets with additional structure, a morphism should consist of maps between vertices, edges, and faces. These maps should be compatible with the inclusion maps η and φ.

Additionally, we want to enforce that the number of vertices in a face does not change under a morphism. E. g. a hexagonal face should not be mapped to a triangular one. This can be described easily with the map $|\cdot|$ from Definition 2.5.4.

Definition 2.5.6. *Let $(V^1, E^1, F^1, \eta^1, \varphi^1)$ and $(V^2, E^2, F^2, \eta^2, \varphi^2)$ be two polygonal complexes. A **polygonal morphism** between them consists of maps*

$$\mu_V : V^1 \to V^2 \qquad \mu_E : E^1 \to E^2 \qquad \mu_F : F^1 \to F^2,$$

with the properties

- *η–**compatible** $v \in \eta^1(e)$ implies $\mu_V(v) \in \eta^2(\mu_E(e))$.*
- *φ–**compatible** $e \in \varphi^1(f)$ implies $\mu_E(e) \in \varphi^2(\mu_F(f))$.*
- ***non–degenerate** For any $f \in F^1$, we have $|f| = |\mu_F(f)|$.*

Corollary 2.5.7. *Let $(\mu_V, \mu_E, \mu_F) : (V^1, E^1, F^1, \eta^1, \varphi^1) \to (V^2, E^2, F^2, \eta^2, \varphi^2)$ be a polygonal morphism. Then, $\eta^1(e) = \{v_1, v_2\}$ implies $\eta^2(\mu_E(e)) = \{\mu_V(v_1), \mu_V(v_2)\}$.*

Corollary 2.5.8. *Let $(\mu_V, \mu_E, \mu_F) : (V^1, E^1, F^1, \eta^1, \varphi^1) \to (V^2, E^2, F^2, \eta^2, \varphi^2)$ be a polygonal morphism. Then, $(\eta^1 \mho \varphi^1)(f) = \{v_1, v_2, \ldots, v_n\}$ implies $(\eta^2 \mho \varphi^2)(\mu_F(f)) = \{\mu_V(v_1), \mu_V(v_2), \ldots, \mu_V(v_n)\}$.*

With the definitions of polygonal complex and polygonal morphism, we can define the categories of polygonal complexes and triangular complexes.

Definition 2.5.9. **PolyComp** *is the category of polygonal complexes, together with polygonal morphisms.*

TriComp *is the category of triangular complexes, together with polygonal morphisms.*

Well–defined. **TriComp** is obtained from **PolyComp** by restriction of objects as in Remark 2.2.4. ☐

2.5.1 Functor PolyComp → TwistPolyComp

Every polygonal complex can be interpreted as a twisted polygonal complex, by subdividing each face barycentrically. In this subsection, we formalise this process as a functor. We start with the set of chambers. For polygonal complexes, the map λ is injective, so we can define a chamber as a triple of incident vertex, edge, and face. This configuration is called a *flag*.

Definition 2.5.10. *Let (V, E, F, η, φ) be a polygonal complex. A* ***flag*** *is an element $(v, e, f) \in V \times E \times F$ with $v \prec e \prec f$.*

The involution σ_0 changes the vertex within a flag. For polygonal complexes, this uniquely defines the involution.

Remark 2.5.11. *Let (V, E, F, η, φ) be a polygonal complex and (v, e, f) be a flag. Then, there exists exactly one other vertex $v^* \in V$ such that (v^*, e, f) is a flag. It satisfies $\eta(e) = \{v, v^*\}$.*

Proof. $\eta(e) = \{v, v^*\}$ for some $v^* \in V$. By definition of incidence, (v^*, e, f) is a flag. ☐

The same thing can be done to construct the involution σ_1 that switches the edge within a flag.

Remark 2.5.12. *Let $P = (V, E, F, \eta, \varphi)$ be a polygonal complex and (v, e, f) be a flag. Then, there exists exactly one other edge $e^* \in E$ such that (v, e^*, f) is a flag.*

Proof. Since P is a polygonal complex, there exist pairwise distinct $v_i \in V$ and $e_i \in E$ with $\varphi(f) = \{e_1, \dots, e_k\}$ and $\eta \between \varphi(f) = \{v_1, \dots, v_k\}$, such that

$$\eta(e_i) = \begin{cases} \{v_i, v_{i+1}\} & 1 \leq i < k \\ \{v_1, v_k\} & i = k. \end{cases}$$

Without loss of generality, we can assume $v = v_1$. Then, only the edges e_1 and e_k are incident to both v and f. ☐

At this point, we can formally state how a polygonal complex has to be formalised as twisted polygonal complex.

Definition 2.5.13. $\mathsf{TwistPoly}$ *is a functor from* **PolyComp** *to* **TwistPolyComp**. *If* $P = (V, E, F, \eta, \varphi)$ *is a polygonal complex,* $\mathsf{TwistPoly}(P)$ *is the twisted polygonal complex* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *with*

- C *is the set of flags.*
- $\lambda : C \to V \times E \times F, \quad (v, e, f) \mapsto (v, e, f)$.
- $\sigma_0 : C \to C$ *maps* (v, e, f) *to the flag* (v^*, e, f) *from Remark 2.5.11.*
- $\sigma_1 : C \to C$ *maps* (v, e, f) *to the flag* (v, e^*, f) *from Remark 2.5.12.*
- $(v, e, f) \sim (v^*, e^*, f^*)$ *if and only if* $v = v^*$ *and* $e = e^*$.

If $\mu = (\mu_V, \mu_E, \mu_F)$ *is a polygonal morphism,* $\mathsf{TwistPoly}(\mu)$ *is the twisted polygonal morphism* $(\mu_V, \mu_E, \mu_F, \mu_C)$, *with* $\mu_C((v, e, f)) := (\mu_V(v), \mu_E(e), \mu_F(f))$.

Well–defined. We have to show that $\mathsf{TwistPoly}(P)$ is a twisted polygonal complex, that $\mathsf{TwistPoly}(\mu)$ is a twisted polygonal map, and that $\mathsf{TwistPoly}$ defines a functor.

1. To show that $\mathsf{TwistPoly}(P)$ is a twisted polygonal complex, we check the conditions of Definition 2.4.1.

   - σ_0 is an involution without fixed points by Remark 2.5.11. By definition,
   $$\lambda_{12}((v, e, f)) = (e, f) = \lambda_{12}((v^*, e, f)) = (\lambda_{12} \circ \sigma_0)(v, e, f).$$
   - σ_1 is an involution without fixed points by Remark 2.5.12. By definition,
   $$\lambda_{02}((v, e, f)) = (v, f) = \lambda_{02}((v, e^*, f)) = (\lambda_{02} \circ \sigma_1)(v, e, f).$$
   - Clearly, $\sim$ is an equivalence relation. By definition, $c_1 \sim c_2$ implies $\lambda_{01}(c_1) = \lambda_{01}(c_2)$.

     Let $(v, e, f) \sim (v, e, f^*)$. We need to show that $\sigma_0(v, e, f) \sim \sigma_0(v, e, f^*)$ holds. The action of σ_0 replaces v by v^* with $\eta(e) = \{v, v^*\}$ in both cases (see Remark 2.5.11). Therefore,
   $$\lambda_{01}(\sigma_0(v, e, f)) = \lambda_{01}(v^*, e, f) = (v^*, e) = \lambda_{01}(v^*, e, f^*) = \lambda_{01}(\sigma_0(v, e, f^*)).$$
   - Consider two chambers (v, e, f) and (v^*, e, f^*). If $v = v^*$, we have $(v, e, f) \sim (v^*, e, f^*)$. Otherwise, $\eta(e) = \{v, v^*\}$. Then, $\sigma_0((v, e, f)) = (v^*, e, f)$, so $(v^*, e, f) \sim (v^*, e, f^*)$.
   - Consider two chambers (v, e, f) and (v^*, e^*, f). Since P is a polygonal complex, there is an alternating sequence $(v_1, e_1, v_2, e_2, \ldots, v_k, e_k)$ (by Definition 2.5.2).

     Now, σ_0 exchanges the tuples (v_i, e_i) and (v_{i+1}, e_i). The involution σ_1 exchanges (v_i, e_i) and (v_i, e_{i-1}). Since both (v, e) and (v^*, e^*) are such tuples and the involutions act transitively on them, we have $(v^*, e^*, f) \in \langle \sigma_0, \sigma_1 \rangle.(v, e, f)$.

2. To show that $\mathsf{TwistPoly}(\mu)$ is a twisted polygonal morphism, we check the conditions of Definition 2.4.5. For that, we set some notation: $\mu : P^1 \to P^2$ with

$$P^k = (V^k, E^k, F^k, \eta^k, \varphi^k)$$
$$\mathsf{TwistPoly}(P^k) = (V^k, E^k, F^k, C^k, \lambda^k, \sigma_0^k, \sigma_1^k).$$

The map $\mu_C : C^1 \to C^2$ is well–defined since polygonal morphisms preserve incidence.

- Since λ^k are defined as identity maps, the compatibility with λ is obvious.
- Compatibility with σ_0: Let $(v, e, f) \in V^1 \times E^1 \times F^1$ with $\eta^1(e) = \{v, v^*\}$. From $\sigma_0^1(v, e, f) = (v^*, e, f)$, we obtain

$$\mu_C(\sigma_0^1(v, e, f)) = (\mu_V(v^*), \mu_E(e), \mu_V(f)).$$

  On the other hand, we can combine $\mu_C(v, e, f) = (\mu_V(v), \mu_E(e), \mu_F(f))$ with $\eta^2(\mu_E(e)) = \{\mu_V(v), \mu_V(v^*)\}$ to conclude

$$\sigma_0^2(\mu_C(v, e, f)) = (\mu_V(v^*), \mu_E(e), \mu_F(f)).$$

- Compatibility with σ_1. Analogous to the compatibility with σ_0.
- Compatibility with $\sim$ is obvious.
- Non–degenerate: By Lemma 2.4.6, it is sufficient to show that the cardinalities of the orbits coincide (they are all finite). Since $|\langle \sigma_0^k, \sigma_1^k \rangle.(v, e, f)| = 2|\varphi^k(f)|$ for any flag $(v, e, f) \in V^k \times E^k \times F^k$, and polygonal morphisms fulfil $|\varphi^1(f)| = |\varphi^2(\mu_F(f))|$, this claim follows.

3. Finally, we have to check the conditions of Definition 2.2.5. But both composition and identities are obviously preserved. □

This functor also preserves the incidence relation, i. e. the incidence relations from Definition 2.5.2 is compatible with the one from Definition 2.4.3.

Remark 2.5.14. *Let $P = (V, E, F, \eta, \varphi)$ be a polygonal complex. Let $x, y \in V \uplus E \uplus F$ with $x \prec y$. Then, $x \prec y$ in* $\mathsf{TwistPoly}(P)$ *as well.*

2.5.2 Edge–Face–Paths and polygonal surfaces

In Section 2.5, we introduced the formalism of polygonal complexes. In this subsection, we restrict polygonal complexes to polygonal surfaces. Like mentioned in Subsection 2.1.1, we have to avoid both edge and vertex ramifications.

Edge ramifications are easy to avoid since we only have to count the number of faces that are incident to a given edge.

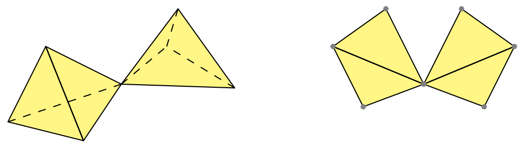
Definition 2.5.15. *Let (V, E, F, η, φ) be a polygonal complex and $e \in E$. It is*

- *an **inner edge** if it is incident to exactly two faces.*
- *a **boundary edge** if it is incident to exactly one face.*
- *a **ramified edge** if it is incident to more than two faces.*

It is easy to see that the definition of edge types for polygonal complexes is compatible with Definition 2.4.10 of edge types for twisted polygonal complexes.

Remark 2.5.16. *Let P be a polygonal complex. An edge is inner/boundary/ramified in P if and only if it is inner/boundary/ramified in* $\mathsf{TwistPoly}(P)$.

In a simplicial surface, there cannot be any ramified edges. Unfortunately, the exclusion of ramified edges is not sufficient to guarantee that a triangular complex is a surface. To see this, imagine two distinct surfaces and identify two vertices (one of each). There are no ramified edges in this construction but there is a vertex which is incident to two "surface patches". We will call these patches *umbrellas*. Visually, this situation could be imagined like this:



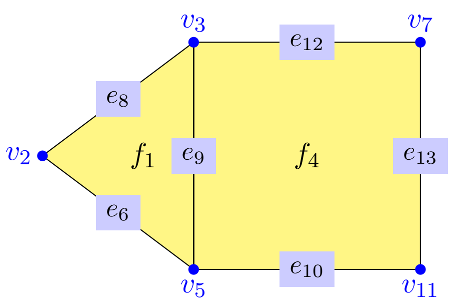
To formally define these patches, we introduce a concept of paths.

Definition 2.5.17. *Let (V, E, F, η, φ) be a polygonal complex. An **edge–face–path** is a sequence $(e_0, f_1, e_1, f_2, \dots, f_n, e_n)$ such that*

- *$e_i \in E$ for all $0 \le i \le n$.*
- *$f_i \in F$ for all $1 \le i \le n$.*
- *e_{i-1} and e_i are incident to f_i for all $1 \le i \le n$.*

*An edge–face–path is called **closed** if $e_0 = e_n$. It is called **non–repeating** if $f_i \neq f_j$ for all $1 \le i < j \le n$. An edge–face–path is called **empty** if $n = 0$.*

Example 2.5.18. *Consider the polygonal complex from Example 2.5.3:*



$(e_8, f_1, e_9, f_4, e_{10})$ *is a non–repeating edge–face–path.*

Two edge–face–paths can be combined if one ends where the other one begins.

Definition 2.5.19. *Let (V, E, F, η, φ) be a polygonal complex. Let $p_1 = (e_0, f_1, \dots, e_n)$ and $p_2 = (e_n, \dots, e_m)$ be two edge–face–paths. Their **path–sum** $p_1 + p_2$ is the edge–face–path $(e_0, \dots, e_n, \dots, e_m)$.*

Definition 2.5.20. *Let (V, E, F, η, φ) be a polygonal complex. An **umbrella–path** is an edge–face–path $(e_0, f_1, e_1, \dots, f_n, e_n)$ such that there is a vertex $v \in V$ that is incident to all edges of the edge–face–path. A non–empty, non–repeating umbrella–path u is called **maximal** if there is no non–empty umbrella path p such that $p + u$ or $u + p$ is a non–repeating umbrella path.*

Remark 2.5.21. *Let (V, E, F, η, φ) be a polygonal complex and $(e_0, f_1, e_1, \dots, f_n, e_n)$ an umbrella–path. Then, there is exactly one vertex incident to all edges e_i.*

Proof. Since u is non–empty, it starts with $(e_0, f_1, e_1, \dots)$. By Definition 2.5.20, we know $\eta(e_0) \cap \eta(e_1) \neq \emptyset$.

If this intersection contains more than one element, we have $\eta(e_0) = \eta(e_1)$. Then, Definition 2.5.2 implies $|\varphi(f_1)| = 2$, in contradiction to $|\varphi(f_1)| \geq 3$. □

We would like to show the following statements for a polygonal complex without ramified edges:

1. For every vertex $v \in V$ and face $f \in F$ with $v \prec f$, there is a unique maximal umbrella–path around v that contains f.
2. For every vertex $v \in V$, the maximal umbrella–paths around v partition the incident faces.

Instead of showing them directly, we make use of the functor TwistPoly that was introduced in Subsection 2.5.1.

We relate the umbrella–paths of the polygonal complex P to the strong umbrella paths of the twisted polygonal complex $\mathsf{TwistPoly}(P)$. Then, we use this relation to carry over the corresponding statements for strong umbrella paths.

Lemma 2.5.22. *Let $P = (V, E, F, \eta, \varphi)$ be a polygonal complex.*

1. *Let $(e_0, f_1, e_1, f_2, e_2, \dots, f_n, e_n)$ be an umbrella–path around v in P. Then,*
$$(c_1^-, c_1^+, c_2^-, c_2^+, \dots, c_n^-, c_n^+)$$
is a strong umbrella path around v in $\mathsf{TwistPoly}(P)$, with $c_i^- := (v, e_{i-1}, f_i)$ and $c_i^+ := (v, e_i, f_i)$.
2. *Let $(c_1, c_2, \dots, c_{2n})$ be a strong umbrella path around v in $\mathsf{TwistPoly}(P)$. Then,*
$$(\lambda_1(c_1), \lambda_2(c_2), \lambda_1(c_3), \lambda_2(c_4), \dots, \lambda_1(c_{2n-1}), \lambda_2(c_{2n}), \lambda_1(c_{2n}))$$
is an umbrella–path around v in P.

3. *These two constructions are inverse to each other.*
4. *Let u_1 and u_2 be two umbrella paths in P, such that $u_1 + u_2$ is also an umbrella path. Then,* $\mathsf{TwistPoly}(u_1 + u_2) = \mathsf{TwistPoly}(u_1) + \mathsf{TwistPoly}(u_2)$.
5. *Let* $\mathsf{TwistPoly}(u_1)$ *and* $\mathsf{TwistPoly}(u_2)$ *be two strong umbrella paths in the twisted polygonal complex* $\mathsf{TwistPoly}(P)$*, such that* $\mathsf{TwistPoly}(u_1)+\mathsf{TwistPoly}(u_2)$ *is a strong umbrella path. Then, $u_1 + u_2$ is a well–defined umbrella path in P.*

Proof.

1. We need to show the properties of Definition 2.4.18. By definition, c_i^- and c_i^+ are 1–adjacent. Since $c_i^+ = (v, e_i, f_i)$ and $c_{i+1}^- = (v, e_i, f_{i+1})$, these two chambers are 2–adjacent.
2. If i is odd, we have $c_{i+1} = \sigma_1(c_i)$, thus $\lambda_{02}(c_{i+1}) = \lambda_{02}(c_i)$. In particular, $\lambda_1(c_i)$ is incident to $\lambda_2(c_{i+1}) = \lambda_2(c_i)$.

   If i is even, we have $c_{i+1} \sim c_i$, thus $\lambda_{01}(c_i) = \lambda_{01}(c_{i+1})$. In particular, $\lambda_1(c_{i+1}) = \lambda_1(c_i)$ is incident to $\lambda_2(c_i)$.

   Finally, $\lambda_1(c_{2n})$ is clearly incident to $\lambda_2(c_{2n})$.
3. This follows from inspection.
4. Clear.
5. Clear. □

With the correspondence between strong umbrella paths and umbrella–paths established, we can now carry over the results of Corollary 2.4.20 and Lemma 2.4.21.

Corollary 2.5.23. *Let (V, E, F, η, φ) be a polygonal complex without ramified edges. Let u be a maximal umbrella–path. Then either u is closed or e_0 and e_n are boundary edges.*

Proof. By Lemma 2.5.22, u is maximal in P if and only if $\mathsf{TwistPoly}(u)$ is maximal in $\mathsf{TwistPoly}(P)$. Thus, Corollary 2.4.20 is applicable. Translating the cases back (with Remark 2.5.16) gives the desired result. □

Lemma 2.5.24. *Let (V, E, F, η, φ) be a polygonal complex with no ramified edges and $v \in V$ such that the number of incident faces is finite. Then, every incident $f \in F$ lies in exactly one maximal umbrella around v (unique up to inversion and cyclic permutation).*

In particular, the maximal umbrellas partition the incident faces.

Proof. We show existence first. There are exactly two flags (v, e_1, f) and (v, e_2, f) that are chambers in $\mathsf{TwistPoly}(P)$ (by Definition 2.5.13).

We apply Lemma 2.4.21 to show that there is exactly one maximal strong umbrella path containing each of them. Since these flags are 1–adjacent, they lie in the same strong umbrella path. We apply Lemma 2.5.22 to construct an umbrella path around v that contains f.

To show uniqueness, assume there are two maximal umbrella–paths around v that contain f. By Lemma 2.5.22, we would have two different maximal strong umbrella paths containing the flags (v, e_1, f) and (v, e_2, f). This contradicts Lemma 2.4.21. □

After defining and characterising umbrella–paths, we can now define vertex ramifications properly: A vertex is ramified if the umbrella partition from Lemma 2.5.24 contains more than one element.

Definition 2.5.25. *Let (V, E, F, η, φ) be a polygonal complex. A vertex $v \in V$ is called*

- ***inner vertex*** *if it is only incident to inner edges and if there is a unique maximal umbrella–path around it.*
- ***boundary vertex*** *if it incident to some boundary edges, no ramified edges and there is a unique maximal umbrella–path around it.*
- ***ramified vertex*** *if it is not incident to a ramified edge and there are at least two maximal umbrella–paths around it.*
- ***chaotic vertex*** *if it is incident to a ramified edge.*

This definition of vertex types for polygonal complexes is fully compatible with Definition 2.4.23 of vertex types for twisted polygonal complexes.

Remark 2.5.26. *Let $P = (V, E, F, \eta, \varphi)$ be a polygonal complex with twisted polygonal complex* $\mathsf{TwistPoly}(P) = (V, E, F, C, \lambda, \sigma_0, \sigma_1)$.
Then, a vertex $v \in V$ is inner/boundary/ramified/chaotic in P if and only if it is inner/boundary/ramified/chaotic in $\mathsf{TwistPoly}(P)$.

With the concepts of edge and vertex ramifications, we can define polygonal surfaces.

Definition 2.5.27. *A polygonal complex (V, E, F, η, φ) is called a* ***polygonal surface*** *if it contains neither ramified edges nor ramified vertices. A triangular complex that is also a polygonal surface is called a* ***simplicial surface****.*

Combining Definition 2.5.9 and Remark 2.2.4 gives the category of polygonal surfaces.

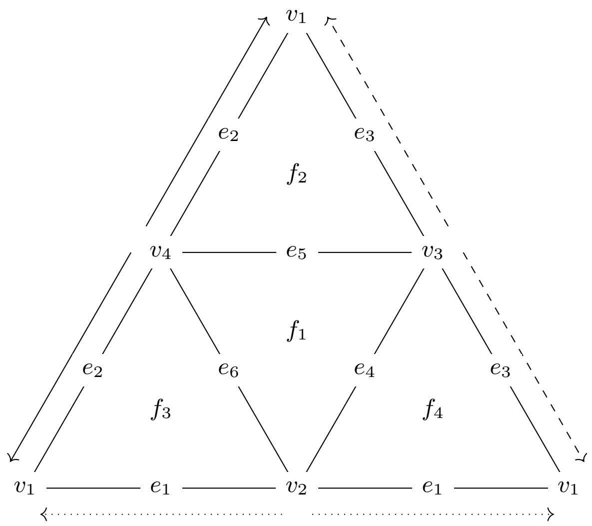
Definition 2.5.28. PolySurf *is the category of polygonal surfaces, together with polygonal morphisms.*

With Remark 2.2.6, the functor TwistPoly can be restricted to polygonal surfaces.

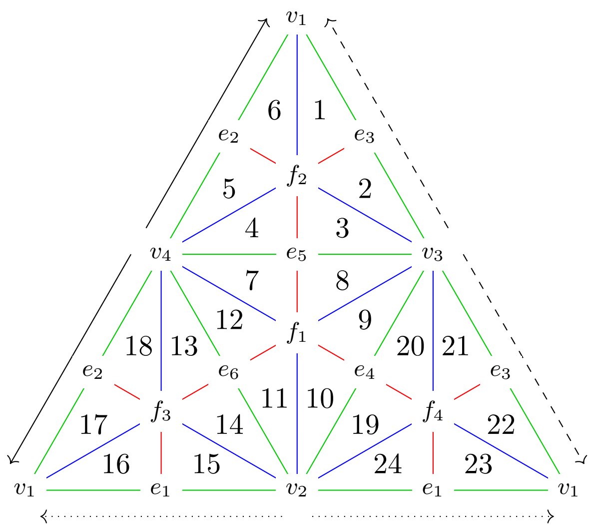
Remark 2.5.29. *Let P be a polygonal complex. It is a polygonal surface if and only if* $\mathsf{TwistPoly}(P)$ *is a twisted polygonal surface.*
In particular, TwistPoly *restricts to a functor* **PolySurf** $\to$ **TwistPolySurf**.

2.6 Category DressSurf of Dress surfaces

The formalisms of polygonal complexes (Section 2.5) and twisted polygonal complexes (Section 2.4) are very combinatorial. Both require extensive work to define a "surface"–concept. The formalism of Dress–surfaces, in contrast, is only applicable to combinatorial surfaces. Consider the tetrahedron:



We can reconstruct the complete incidence structure from the barycentric subdivision, if we know the adjacency relations.



We can encode these adjacencies by involutions, e. g.

$$\sigma_0 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$$
$$\sigma_1 = (1,6)(2,3)(4,5)(7,12)(8,9)(10,11)(13,18)(14,15)(16,17)(19,24)(20,21)(22,23)$$
$$\sigma_2 = (1,22)(2,21)(3,8)(4,7)(5,18)(6,17)(9,20)(10,19)(11,14)(12,13)(15,24)(16,23)$$

The vertices of the combinatorial surface stand in bijection to the orbits of $\langle \sigma_1, \sigma_2 \rangle$, the edges to the orbits of $\langle \sigma_0, \sigma_2 \rangle$, and the faces to the orbits of $\langle \sigma_0, \sigma_1 \rangle$.

This formalisation of surfaces already appeared in [28] and [5].

Definition 2.6.1. *A **Dress–surface** is a quadruple* $(C, \sigma_0, \sigma_1, \sigma_2)$ *with*

- *A set C called **chambers**.*

- *Three involutions $\sigma_k : C \to C$.*

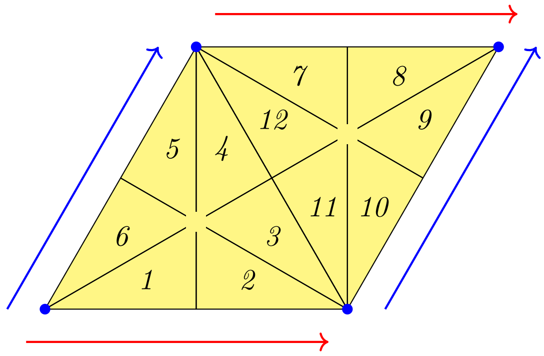
- *σ_0 and σ_1 do not fix any chamber.*

- *$(\sigma_0 \sigma_2)^2 = id_C$.*

Example 2.6.2. *The two–torus $(\{1, \ldots, 12\}, \sigma_0, \sigma_1, \sigma_2)$ with*

$$\begin{aligned}
\sigma_0 &= (1,2)(3,4)(5,6)(7,8)(9,10)(11,12) \\
\sigma_1 &= (1,6)(2,3)(4,5)(7,12)(8,9)(10,11) \\
\sigma_2 &= (1,7)(2,8)(3,11)(4,12)(5,9)(6,10),
\end{aligned}$$

illustrated by



is a Dress–surface.

Example 2.6.2 also illustrates that two vertices within a face can be identical.

As stated in the beginning of Section 2.6, the vertices, edges, and faces of the combinatorial surface can be described purely in terms of the involutions:

Definition 2.6.3. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface.*

- *The orbits of $\langle \sigma_1, \sigma_2 \rangle$ on C are called **vertices**.*

- *The orbits of $\langle \sigma_0, \sigma_2 \rangle$ on C are called **edges**.*

- *The orbits of $\langle \sigma_0, \sigma_1 \rangle$ on C are called **faces**.*

We can define a transitive incidence relation on vertices, edges, and faces. Intuitively, a vertex is incident to an edge if they lie in the same chamber. In the formalism of Dress–surfaces, this chamber is contained in the sets representing vertex and edge.

Definition 2.6.4. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface. We define an **incidence** relation $\prec$ as follows:*

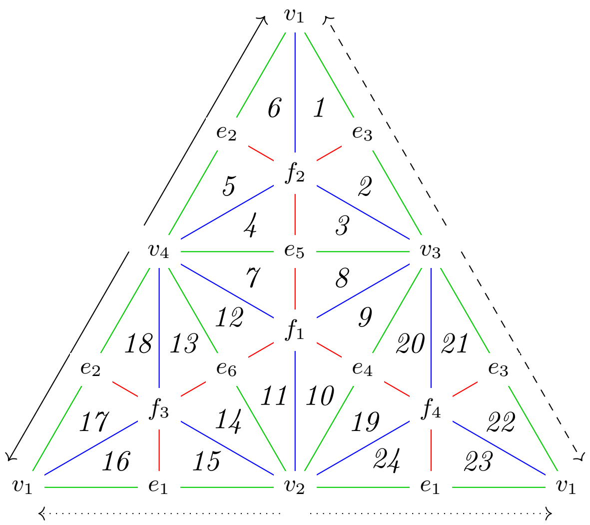
- *A vertex x is incident to an edge y if $x \cap y \neq \emptyset$.*

- *A vertex x is incident to a face y if $x \cap y \neq \emptyset$.*
- *An edge x is incident to a face y if $x \cap y \neq \emptyset$.*

Example 2.6.5. *Consider the tetrahedron $(\{1,\dots,12\},\sigma_0,\sigma_1,\sigma_2)$ with*

$$\sigma_0 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$$
$$\sigma_1 = (1,6)(2,3)(4,5)(7,12)(8,9)(10,11)(13,18)(14,15)(16,17)(19,24)(20,21)(22,23)$$
$$\sigma_2 = (1,22)(2,21)(3,8)(4,7)(5,18)(6,17)(9,20)(10,19)(11,14)(12,13)(15,24)(16,23)$$

from the start of Section 2.6:



The vertices are the orbits of $\langle\sigma_1,\sigma_2\rangle$ on $\{1,\dots,12\}$. With the labels from the picture:

$$v_1 = \{1,6,16,17,22,23\} \qquad v_2 = \{10,11,14,15,19,24\}$$
$$v_3 = \{2,3,8,9,20,21\} \qquad v_4 = \{4,5,7,12,13,18\}$$

The edges are the orbits of $\langle\sigma_0,\sigma_2\rangle$ on $\{1,\dots,12\}$. With the labels from the picture:

$$e_1 = \{15,16,23,24\} \qquad e_2 = \{17,18,21,22\} \qquad e_3 = \{1,2,5,6\}$$
$$e_4 = \{9,10,19,20\} \qquad e_5 = \{3,4,7,8\} \qquad e_6 = \{11,12,13,14\}$$

The faces are the orbits of $\langle\sigma_0,\sigma_1\rangle$ on $\{1,\dots,12\}$. With the labels from the picture:

$$f_1 = \{7,8,9,10,11,12\} \qquad f_2 = \{1,2,3,4,5,6\}$$
$$f_3 = \{13,14,15,16,17,18\} \qquad f_4 = \{19,20,21,22,23,24\}$$

The vertex v_2 is incident to the edge e_6 since $v_2 \cap e_6 = \{11,14\}$, but is not incident to the edge e_5 since $v_2 \cap e_5 = \emptyset$.

Remark 2.6.6. *Let $(C,\sigma_0,\sigma_1,\sigma_2)$ be a Dress–surface. Incidence is a transitive relation.*

Proof. Let x be a vertex, y be an edge, and z be a face, such that $x \cap y \neq \emptyset$ and $y \cap z \neq \emptyset$. We want to show that $x \cap z \neq \emptyset$ as well. Define a group epimorphism by

$$\langle a, b \mid a^2, b^2, (ab)^2 \rangle \to \langle \sigma_0, \sigma_2 \rangle, \qquad a \mapsto \sigma_0 \qquad b \mapsto \sigma_2.$$

Thus, if c lies in the orbit y of $\langle \sigma_0, \sigma_2 \rangle$, we have $y = \{c, \sigma_0(c), \sigma_2(c), \sigma_0\sigma_2(c)\}$.

Since x is an orbit of $\langle \sigma_1, \sigma_2 \rangle$, the set $x \cap y$ contains $\{c, \sigma_2(c)\}$ or $\{\sigma_0(c), \sigma_2\sigma_0(c)\}$.

Since z is an orbit of $\langle \sigma_0, \sigma_1 \rangle$, the set $y \cap z$ contains $\{c, \sigma_0(c)\}$ or $\{\sigma_2(c), \sigma_0\sigma_2(c)\}$. In all four possible combinations, there is a non–trivial intersection between x and z. $\square$

To define polygonal surfaces and twisted polygonal surfaces, we had to exclude vertex and edge ramifications (compare Subsection 2.1.1). For Dress–surfaces, this is not necessary: By construction of the vertices and edges, it is impossible for ramifications to occur. We can still distinguish whether vertices and edges lie in the inner part of the surface or on the boundary.

Definition 2.6.7. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface. An edge y is called*

- ***inner edge*** *if $|y| = 4$.*
- ***boundary edge*** *if $|y| = 2$.*

A vertex x is called

- ***inner vertex*** *if it is only incident to inner edges.*
- ***boundary vertex*** *if it is incident to at least one boundary vertex.*

Next, we define morphisms for Dress–surfaces. Clearly, they should map the chambers onto each other and preserve the involutions. In addition, a face should keep its size, e. g. a hexagonal face should not be mapped to a triangular one.

Definition 2.6.8. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ and $(D, \tau_0, \tau_1, \tau_2)$ be two Dress–surfaces. A **Dress morphism** is a map $\mu : C \to D$ satisfying*

- ***compatible with*** *σ_k: $\mu \circ \sigma_k = \tau_k \circ \mu$ for all $k \in \{0, 1, 2\}$.*
- ***non–degenerate****: The restriction $\mu : \langle \sigma_0, \sigma_1 \rangle.c \to \langle \tau_0, \tau_1 \rangle.\mu(c)$ is bijective.*

*If μ is bijective, it is called a **Dress isomorphism**. If $(C, \sigma_0, \sigma_1, \sigma_2) = (D, \tau_0, \tau_1, \tau_2)$ and μ is bijective, it is called a **Dress automorphism**.*

Definition 2.6.9. DressSurf *is the category of Dress surfaces, together with Dress morphisms.*

We can also define coverings of Dress–surfaces, if we extend the non–degeneracy condition of Definition 2.6.8 to all subgroups generated by two involutions. This enforces three things:

- Inner edges are mapped to inner edges, and boundary edges are mapped to boundary edges.
- Inner vertices are mapped to inner vertices, and boundary vertices are mapped to boundary vertices.
- A vertex contained in k chambers is mapped to a vertex with k chambers.

Definition 2.6.10. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ and $(D, \tau_0, \tau_1, \tau_2)$ be two Dress–surfaces. A **Dress covering morphism** is a Dress morphism $\mu : C \to D$ where the restrictions*

$$\mu : \langle \sigma_0, \sigma_2 \rangle.c \to \langle \tau_0, \tau_2 \rangle.\mu(c)$$
$$\mu : \langle \sigma_1, \sigma_2 \rangle.c \to \langle \tau_1, \tau_2 \rangle.\mu(c)$$

are bijective for each $c \in C$.

2.7 Functors

In the previous sections, we introduced several different formalisations of combinatorial surfaces. In this section, we look closer at their relations to each other.

2.7.1 Functors between TriComp and SimpComp^2

In this subsection, we explore the connections between simplicial complexes and triangular complexes. We start by showing that every simplicial complex can be interpreted as a triangular complex.

Definition 2.7.1. Poly *is a functor from* **SimpComp**^{2} *to* **TriComp**. *It maps the simplicial complex (V, Δ) to the triangular complex (V, E, F, id_E, id_F), with*

$$E := \{x \in \Delta \mid |x| = 2\}$$
$$F := \{x \in \Delta \mid |x| = 3\}$$

and the simplicial morphism μ_V to (μ_V, μ_E, μ_F), where μ_E and μ_F are defined as element–wise application of μ_V.

Well–defined. We have to show that (V, E, F, id_E, id_F) is a triangular complex:

1. For any $f \in F$ we have $f = \{v_1, v_2, v_3\} \subseteq V$. The sequence

$$(v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, \{v_1, v_3\})$$

   satisfies the requirements of Definition 2.5.2.

2. Since (V, Δ) is homogeneous, all vertices and edges lie in a face. Since all subsets of a simplex are also simplices, every vertex thus lies in an edge.

Next, we have to show that (μ_V, μ_E, μ_F) is a polygonal morphism. This follows immediately from the construction.

The functorial properties (Definition 2.2.5) are obvious. $\square$

We are also interested in the converse question: Which triangular complexes can be interpreted as simplicial complexes? To answer it, we construct the "obvious" functor from **TriComp** to **SimpComp**^{2}: We map (V, E, F, η, φ) to the simplicial complex that we obtain when we forget the different names of edges and faces, and just keep the set of incident vertices.

Definition 2.7.2. Simp *is a functor from* **TriComp** *to* **SimpComp**^{2}. *It maps the triangular complex* (V, E, F, η, φ) *to the simplicial complex* (V, Δ) *with*

$$\Delta := \{\{v\} \mid v \in V\} \cup \{\eta(e) \mid e \in E\} \cup \{(\eta \between \varphi)(f) \mid f \in F\},$$

and the polygonal morphism (μ_V, μ_E, μ_F) *to* μ_V.

Well–defined. First, we show that (V, Δ) is a simplicial complex: Consider the different sets in Δ in turn.

- The set $\{v\}$ has no non–empty subset except $\{v\}$.
- The set $\eta(e)$ consists of two vertices in V. Since both of these appear as singletons in Δ, there is no problem here.
- The set $(\eta \between \varphi)(f) = \bigcup_{e \in \varphi(f)} \eta(e)$ consists of three elements of V by Definition 2.5.2. The one–element–subsets are $\{v\} \in \Delta$. Consider a two–element–subset $\{v, w\}$. Since $\eta_{|\varphi(f)}$ is injective and $|\varphi(f)| = |\operatorname{Pot}_2(\{1,2,3\})| = 3$, there is an edge $e \in \varphi(f)$ with $\eta(e) = \{v, w\}$.

Since every vertex and edge of a polygonal complex is contained in a face, (V, Δ) is homogeneous of dimension 2.

Next, we show that μ_V is a simplicial morphism. Consider a simplex $X \in \Delta$.

1. If $X = \{v\}$, then $\{\mu(v)\}$ is a simplex of the image.
2. If $X = \{v_1, v_2\}$, the claim follows from Corollary 2.5.7.
3. If $X = \{v_1, v_2, v_3\}$, the claim follows from Corollary 2.5.8.

The functorial properties (Definition 2.2.5) are obvious. $\square$

The functor Simp inverts the functor Poly:

Remark 2.7.3. *Let S be a homogeneous simplicial complex of dimension 2. Then,* $\mathsf{Simp}(\mathsf{Poly}(S)) = S$.

Unfortunately, the opposite inversion $\mathsf{Poly}(\mathsf{Simp}(P))$ is in general not isomorphic to P (isomorphism is the most we can hope for in this situation since the functor Simp forgets the previous labels of edges and faces). If we consider Definition 2.7.2 in detail, the reason becomes clear: If two edges (or faces) share the same vertices, they are collapsed into one edge (or face). This motivates the following definition.

Definition 2.7.4. *A triangular complex (V, E, F, η, φ) is called **vertex–faithful** if η and $\eta \between \varphi$ are injective.*

Vertex–faithful triangular complexes are uniquely determined by the vertex–edge and the vertex–face incidence relations.

Lemma 2.7.5. *Let V, E, and F be sets with maps $\eta : E \to \mathrm{Pot}_2(V)$ and $\psi : F \to \mathrm{Pot}_3(V)$ with the following properties:*

1. Vertex–faithful: *η and ψ are injective.*
2. Every vertex lies in an edge: *For every $v \in V$, there is an $e \in E$ with $v \in \eta(e)$.*
3. Every edge lies in a face: *For every $e \in E$, there is an $f \in F$ with $\eta(e) \subseteq \psi(f)$.*
4. Every face has three edges: *For every $f \in F$ and every two–element subset $S \subseteq \psi(f)$, there is an $e \in E$ with $\eta(e) = S$.*

Then the map

$$\varphi : F \to \mathrm{Pot}_3(E) \qquad f \mapsto \{e \in E \mid \eta(e) \subseteq \psi(f)\} \tag{2.1}$$

is well–defined and (V, E, F, η, φ) is a vertex–faithful triangular complex.

Proof. We start by showing that φ is well–defined. For $f \in F$, the set $\psi(f)$ consists of three elements. Since $|\eta(e)| = 2$ for all $e \in E$ and η is injective, there can be at most three edges e that fulfil $\eta(e) \subseteq \psi(f)$. By our fourth assumption, there are also at least three.

Now we need to show that (V, E, F, η, φ) is a triangular complex. By construction, every vertex lies in an edge and every edge lies in a face. Now consider the triangle–condition. By definition of ψ, we have for $f \in F$

$$\bigcup_{e \in \varphi(f)} \eta(e) = \psi(f), \tag{2.2}$$

which has three elements. Call $\varphi(f) = \{e_1, e_2, e_3\}$. Since η is injective, we can find a labelling $\{v_1, v_2, v_3\}$ of the vertices in $\psi(f)$ such that $\eta(e_1) = \{v_1, v_2\}$, $\eta(e_2) = \{v_2, v_3\}$, and $\eta(e_3) = \{v_1, v_3\}$.

The injectivity–assumption suffices to make this triangular complex vertex–faithful. $\square$

For vertex–faithful triangular complexes, the functor Poly inverts the functor Simp.

Remark 2.7.6. *Let P be a vertex–faithful triangular complex. Then, $\mathsf{Poly}(\mathsf{Simp}(P))$ is isomorphic to P.*

Proof. Let $P = (V, E, F, \eta, \varphi)$, then $\mathsf{Simp}(P) = (V, \Delta_0 \cup \Delta_1 \cup \Delta_2)$, with

$$\Delta_0 = \{\{v\} \mid v \in V\}, \qquad \Delta_1 = \{\eta(e) \mid e \in E\}, \qquad \Delta_2 = \{(\eta \mho \varphi)(f) \mid f \in F\}.$$

Since P is vertex–faithful, $\eta : E \to \Delta_1$ and $(\eta \mho \varphi) : F \to \Delta_2$ are bijections. Now, $\mathsf{Poly}(\mathsf{Simp}(P)) = (V, \Delta_1, \Delta_2, id_{\Delta_1}, id_{\Delta_2})$ and $(id_V, \eta, \eta \mho \varphi)$ is a polygonal isomorphism from P to $\mathsf{Poly}(\mathsf{Simp}(P))$. $\square$

Corollary 2.7.7. *Let (V, Δ) be a homogeneous simplicial complex of dimension 2. Then, there is (up to isomorphism) exactly one vertex–faithful triangular complex P with $\mathsf{Simp}(P) = (V, \Delta)$.*

Proof. From Remark 2.7.3, we obtain the existence of such a P.

If P_1 and P_2 are two vertex–faithful triangular complexes with $\mathsf{Simp}(P_1) = \mathsf{Simp}(P_2)$, Remark 2.7.6 implies that P_1 is isomorphic to P_2. $\square$

At this point, Remark 2.7.6 and Remark 2.7.3 introduce a very deep connection between homogeneous simplicial complexes of dimension 2 and vertex–faithful triangular complexes (in fact, it is an equivalence of categories).

For simplicial complexes, we introduced the notions of shadow and twilight morphism (Definition 2.3.2). It stands to reason that we can transfer these concepts to vertex–faithful triangular complexes as well.

Definition 2.7.8. *Let $P_1 = (V_1, E_1, F_1, \eta_1, \varphi_1)$ and $P_2 = (V_2, E_2, F_2, \eta_2, \varphi_2)$ be two vertex–faithful triangular complexes. A triple (μ_V, μ_E, μ_F) of morphisms*

$$\mu_V : V_1 \to V_2 \qquad\qquad \mu_E : E_1 \to E_2 \qquad\qquad \mu_F : F_1 \to F_2$$

is called ***polygonal shadow morphism*** *if all $x \in \mathrm{Pot}(V_1)$ with the properties*

1. *$x \neq \{v\}$ for all $v \in V_1$,*
2. *$x \neq \eta_1(e)$ for all $e \in E_1$,*
3. *$x \neq (\eta_1 \mho \varphi_1)(f)$ for all $f \in F_1$,*

also satisfy

1. *$Y \neq \{v\}$ for all $v \in V_2$,*
2. *$Y \neq \eta_2(e)$ for all $e \in E_2$,*
3. *$Y \neq (\eta_2 \mho \varphi_2)(f)$ for all $f \in F_2$,*

for $Y := \{\mu_V(y) \mid y \in x\}$. It is called ***polygonal twilight morphism*** *if it is both a polygonal morphism and a polygonal shadow morphism.*

The functor Simp preserves shadow and twilight morphisms.

Lemma 2.7.9. *Let $(\mu_V, \mu_E, \mu_F) : P^1 \to P^2$ be a polygonal shadow morphism between the triangular complexes P^1 and P^2. Then, μ_V is a simplicial shadow morphism between* $\mathsf{Simp}(P^1)$ *and* $\mathsf{Simp}(P^2)$.

Proof. We use the names

$$P^k = (V^k, E^k, F^k, \eta^k, \varphi^k) \qquad\qquad \mathsf{Simp}(P^k) = (V^k, \Delta^k).$$

Let $x \in \mathrm{Pot}(V^1)\backslash\Delta^1$. Since $x \notin \Delta^1$ and Δ^1 is given by elements of V^1, images of η^1 and images of φ^1, Definition 2.7.8 is applicable. We deduce $\{\mu_V(y) \mid y \in x\} \notin \Delta_2$. $\square$

2.7.2 Functor DressSurf $\to$ TwistPolyComp

In Section 2.4, the formalism of twisted polygonal complexes is introduced. The formalism of Dress–surfaces is introduced in Section 2.6. In this subsection, we explain in which sense a Dress–surface can be interpreted as a twisted polygonal surface.

Definition 2.7.10. TwistDress *is a functor from* **DressSurf** *to* **TwistPolyComp**. *If $P = (C, \sigma_0, \sigma_1, \sigma_2)$ is a Dress surface,* TwistDress(P) *is the twisted polygonal surface $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ with*

- *V is the set of vertices (orbits of $\langle\sigma_1, \sigma_2\rangle$).*
- *E is the set of edges (orbits of $\langle\sigma_0, \sigma_2\rangle$).*
- *F is the set of faces (orbits of $\langle\sigma_0, \sigma_1\rangle$).*
- *$\lambda : C \to V \times E \times F$ maps each chamber c to the orbits in which it lies.*
- *The equivalence classes of $\sim$ are defined as $[c]_\sim := \{c, \sigma_2(c)\}$.*

If μ is a Dress morphism, TwistDress$(\mu) = (\mu_V, \mu_E, \mu_F, \mu)$, *with μ_V, μ_E, and μ_F defined element–wise, is a twisted polygonal edge–covering.*

Well–defined. We have to check several things in turn.

- Prove that TwistDress(P) is a twisted polygonal complex, by checking the conditions of Definition 2.4.1.
    1. By Definition 2.6.1, σ_0 and σ_1 are involutions without fixed points.
    2. Consider $c_1, c_2 \in C$ with $c_1 \neq c_2$ and $c_1 \sim c_2$. This is only possible if $c_2 = \sigma_2(c_1)$. Since vertices are the orbits of $\langle\sigma_1, \sigma_2\rangle$ and edges are the orbits of $\langle\sigma_0, \sigma_2\rangle$, we have $\lambda_{01}(c_1) = \lambda_{01}(\sigma_2(c_1))$.

       By Definition 2.6.1, $(\sigma_0\sigma_2)^2$ is the identity, so we also have

       $$\sigma_2(\sigma_0(c_1)) = \sigma_0(\sigma_2(c_1)) = \sigma_0(c_2),$$

       so $\sigma_0(c_1) \sim \sigma_0(c_2)$.

3. Let c_1 and c_2 be two chambers with $\langle\sigma_0,\sigma_2\rangle.c_1 = \langle\sigma_0,\sigma_2\rangle.c_2$. This implies $c_2 \in \{c_1, \sigma_2(c_1), \sigma_0(c_1), \sigma_2\sigma_0(c_1)\}$. In the first two cases we have $c_2 \sim c_1$, in the last two we have $c_2 \sim \sigma_0(c_1)$.
4. Let c_1 and c_2 be two chambers with $\langle\sigma_0,\sigma_1\rangle.c_1 = \langle\sigma_0,\sigma_1\rangle.c_2$. Then, $c_1 \in \langle\sigma_0,\sigma_1\rangle.c_2$.

- Prove that $\mathsf{TwistDress}(P)$ is a twisted polygonal surface, by checking the conditions of Definition 2.4.24.

  Since all edges of a Dress surface consists of at most 4 chambers, the twisted polygonal complex cannot have any boundary edges (compare Definition 2.4.10).

  Since the strong umbrella paths correspond to the orbits of $\langle\sigma_1,\sigma_2\rangle$, there are no ramified vertices as well.

- Prove that $(\mu_V, \mu_E, \mu_F, \mu)$ is a twisted polygonal morphism by checking the conditions of Definition 2.4.5.

  The λ–compatibility follows from the definition of λ. The other conditions follow directly from Definition 2.6.8.

Since the functorial properties are obvious, this completes the proof. □

Conversely, any twisted polygonal surface can be interpreted as a Dress–surface. To formalise this, we define the involution σ_2 for twisted polygonal surfaces.

Remark 2.7.11. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal surface. Then,*

$$\sigma_2 : C \to C \qquad\qquad c \mapsto \begin{cases} c & [c]_\sim = \{c\} \\ \hat{c} & [c]_\sim = \{c, \hat{c}\} \end{cases}$$

is an involution.

With the preparation of Remark 2.7.11, any twisted polygonal surface can be described as a Dress–surface. We take care to mention that not every twisted polygonal morphism can be described by a Dress–morphism.

Lemma 2.7.12. *Let $S = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal surface.*
Then, the Dress–surface $T = (C, \sigma_0, \sigma_1, \sigma_2)$ with σ_2 from Remark 2.7.11 satisfies that $\mathsf{TwistDress}(T)$ *is isomorphic to S.*

Proof. In a twisted polygonal surface, the sets of vertices and edges can be reconstructed (up to bijection) from $(C, \sigma_0, \sigma_1, \sim)$. □

The functor $\mathsf{TwistDress}$ preserves the sets of vertices, edges, and faces, together with the incidence relation.

Remark 2.7.13. *Let S be a Dress–surface. There are canonical bijections between the sets of vertices, edges, and faces of S, and those of* $\mathsf{TwistDress}(S)$*. In addition, these bijections preserve the incidence relation.*

Proof. Since the vertices, edges, and faces of $\mathsf{TwistDress}(S)$ are defined by those of S, they stand in bijection.

Consider the incidence relation and let x, y be vertices, edges, or faces. We have $x \prec y$ in S if and only if $x \cap y \neq \emptyset$. This is the case if and only if there is a chamber $c \in C$ with $c \in x \cap y$. By definition of λ in $\mathsf{TwistDress}(S)$, the sets x and y will be components of $\lambda(c)$. By Definition 2.4.3, they are incident in S if and only if they are incident in $\mathsf{TwistDress}(S)$. □

2.8 Combinatorial complexes and surfaces

In the previous sections, we described several different models for combinatorial surfaces. They all have different advantages and drawbacks that make them more or less suitable to certain applications.

- To model very regular combinatorial surfaces, the formalism of Dress–surfaces is usually preferable since it makes it easy to employ group–theoretic arguments.
- To study modifications of combinatorial surfaces, where we just change a small part of the surface, the model of polygonal surfaces might be preferable since it makes the description of purely local changes easier.

However, there are also situations where we would like to switch between different models or where we do not care about the specific model that we use. The first situation can be partially resolved by using the appropriate functors, but the second one points to a more general criticism: The models so far all claim to describe “combinatorial surfaces”, yet are clearly distinct. Ideally, we would have a nice, coherent characterisation to explain what a “combinatorial surface” is.

Unfortunately, we do not have this characterisation. This section aims to take a step in that direction, without claiming to solve the problem. It gives a workable criterion to decide which properties can be generalised over all models and might give some direction to future research.

Definition 2.8.1. *A* ***combinatorial complex*** *is either a twisted polygonal complex, a polygonal complex, or a Dress–surface.*

A ***combinatorial surface*** *is either a twisted polygonal surface, a polygonal surface, or a Dress–surface.*

So far, these are just names that hide the specific formalisation. To go further, we have to define which properties can be generalised. For example, all models so far allow us to talk about “vertices”, “edges”, and “faces”. Also, all of them allow us to ask the question whether a given edge is an “inner” or a “boundary” edge.

Of course, we cannot pick any random definition here. The conceptions of “inner edges” should be compatible between the models. Since we describe the models as categories, the compatibility between models is described by functors. Thus, we search for properties that do not change if we apply the functors TwistDress and TwistPoly.

Definition 2.8.2. *A **combinatorial property** is a property that is defined for each possible combinatorial complex (or surface) such that:*

- *A polygonal complex/surface P has the property if and only if* $\mathsf{TwistPoly}(P)$ *has the property.*
- *A Dress–surface S has the property if and only if* $\mathsf{TwistDress}(S)$ *has the property.*

We give a rough intuition about which properties one can expect to be combinatorial: If one can formulate the property for a concrete example, without relying on the specifics of the objects, it usually turns out to be a combinatorial property.

Example 2.8.3. *The notion of "combinatorial surface" in Definition 2.8.1 is a combinatorial property: It is defined in Definition 2.4.24 for twisted polygonal complexes and in Definition 2.5.27 for polygonal complexes. By Remark 2.5.29 and Definition 2.7.10, it is preserved under the functors.*

2.8.1 Basic properties

In Section 2.8, we introduced the general description of combinatorial complexes, together with the crucial notion of combinatorial properties (Definition 2.8.2). In this subsection, we give a few examples of some basic combinatorial properties.

Remark 2.8.4. *The sets of vertices, edges, and faces, together with the transitive incidence relation, are combinatorial properties.*

Proof. Vertices, edges, and faces can be defined for twisted polygonal complexes (Definition 2.4.1), polygonal complexes (Definition 2.5.2), and Dress–surfaces (Definition 2.6.3). The functors TwistPoly and TwistDress induce bijections between the corresponding sets of different models.

The incidence relation is defined for twisted polygonal complexes (Definition 2.4.3), polygonal complexes (Definition 2.5.2), and Dress–surfaces (Definition 2.6.4). The functor TwistPoly preserves incidence by Remark 2.5.14. The functor TwistDress preserves incidence by Remark 2.7.13. ☐

Remark 2.8.5. *The types of edges (inner, boundary, ramified) and vertices (inner, boundary, ramified, chaotic) are combinatorial properties.*

Proof. Definitions of vertex types: 2.4.23, 2.5.25, 2.6.7.

Definition of edge types: 2.4.10, 2.5.15, 2.6.7.

The compatibility is shown in Remark 2.5.26 and Remark 2.5.16 for TwistPoly. For TwistDress, it follows from the observation that inner/boundary edges are mapped to inner/boundary edges, as well as inner/boundary vertices. ☐

At this point, the strength of the combinatorial description becomes clear: Since we can talk about "inner edges" independently from the concrete model, we can use this concept in further definitions.

Definition 2.8.6. *A combinatorial surface is* ***closed*** *if all of its edges are inner edges.*

Well–defined. By Remark 2.8.4 and Remark 2.8.5, all terms of the definition are combinatorial properties. □

Definition 2.8.7. *Let C be a combinatorial complex, such that V is the set of vertices, E the set of edges, and F the set of faces. The number $\chi(C) := |V| - |E| + |F|$ is called the* ***Euler–characteristic*** *of the surface.*

Well–defined. By Remark 2.8.4, this is a combinatorial property. □

Definition 2.8.8. *Let C be a combinatorial surface. It is called* ***spherical*** *if $\chi(C) = 2$.*

Well–defined. This definition only requires the `Euler`–characteristic, which is defined for all combinatorial surfaces in Definition 2.8.7. □

Finally, we define *triangular* for twisted polygonal complexes and Dress–surfaces. Not surprisingly, it also is a combinatorial property.

Definition 2.8.9. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex. We define the map*

$$|\cdot| : F \to \mathbb{N} \qquad\qquad f \mapsto \frac{1}{2}|\lambda_2^{-1}(f)|.$$

Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface with set of faces F. We define the map

$$|\cdot| : F \to \mathbb{N} \qquad\qquad f \mapsto \frac{1}{2}|f|$$

A combinatorial complex is called ***triangular*** *if $|f| = 3$ for each face f.*

Well–defined. It suffices to show that $|\cdot|$ is a combinatorial property. It is defined for polygonal complexes in Definition 2.5.4.

Consider the functor TwistPoly. The face f with $|f|$ incident vertices gives rise to $2|f|$ chambers c with $\lambda_2(c) = f$. Therefore, TwistPoly preserves $|\cdot|$.

Consider the functor TwistDress. A face is an orbit under $\langle\sigma_0, \sigma_1\rangle$. Each element is mapped to the orbit under λ_2. Thus, $|\cdot|$ is preserved by TwistDress. □

2.9 When to use which formalism?

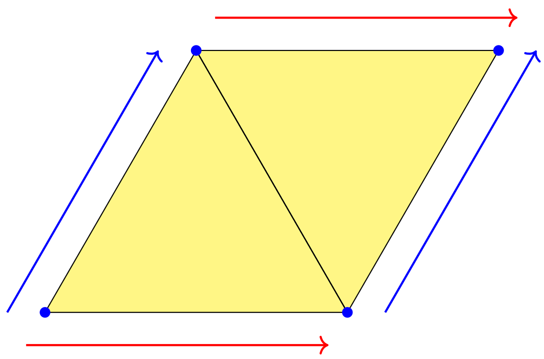
After reading (or skimming) this chapter, a question remains: Which of these formalisations for combinatorial complexes should be used? As Section 2.8 alludes to, most properties can be found in all of them.

As shown in Subsection 2.5.1 and Subsection 2.7.2, twisted polygonal complexes are the most general formalism. This would make them the natural choice, except for the fact that the formalism of twisted polygonal complexes is much more complicated than the other formalisms. Thus, one has to decide whether one needs the full generality.

Otherwise, one usually gains from restricting attention to a smaller class of combinatorial complexes.

Lemma 2.7.12 shows that every twisted polygonal surface can be described by a Dress–surface. Therefore, if one is only interested in surfaces, one can use the formalism of Dress–surfaces. It is also more practical for group–theoretic considerations (which we use in Chapter 9).

In contrast, the formalism of polygonal complexes is more combinatorial. It allows the flexibility of ramified vertices and edges, but it restricts the shape of possible faces. Intuitively, the vertices and edges incident to a face have to be distinct (none is incident twice). Thus, surfaces like the torus



from the start of Section 2.4 cannot be modelled by polygonal complexes.

We can restrict the formalism of polygonal complexes even further if we demand that they are *triangular and vertex–faithful*. By Remark 2.7.6, this allows us to describe the combinatorial complexes by simplicial complexes. Their combinatorial structure is much simpler (and more homogeneous) than that of polygonal complexes.

3 Graph properties

Combinatorial complexes and surfaces can be interpreted and analysed in many different ways. This chapter focuses on the different graphs that can be found within a combinatorial complex or surface. To do so, Section 3.1 contains a short introduction to the concepts of graph theory that we will use, along with our notation for graphs.

In this chapter, we discuss three different graphs:

- In Section 3.2, we work with the *vertex–edge–graph*. This is a graph formed from the vertices and edges of a combinatorial complex.

  In this context, we discuss vertex–colourings of twisted polygonal complexes.

- In Section 3.3, we work with the *face–edge–graph*. This is a graph formed from the faces and edges of a combinatorial surface.

  In this context, we discuss edge–colourings and face–colourings of twisted triangular surfaces.

- In Section 3.4, we work with the *boundary graph*. This is a graph formed from the boundary vertices and boundary edges of a combinatorial surface.

3.1 Basic graph definitions

This section defines the basic notation for the graphs that we use. The interested reader can find an extensive treatment of many different aspects of graph theory in [35].

There are several different definitions for graphs in the literature. We choose a definition that is compatible with the notation for polygonal complexes from Definition 2.5.2.

Essentially, a graph consists of vertices and edges, such that every edge is incident to exactly two vertices.

Definition 3.1.1. *A* ***graph*** *is a triple* (V, E, η) *where* V *is a set called* ***vertices****,* E *is a set called* ***edges****, and* $\eta : E \to \mathrm{Pot}_2(V)$ *is a map.*
A vertex $v \in V$ *is called* ***incident*** *to an edge* $e \in E$ *if* $v \in \eta(e)$*.*
If $V = E = \emptyset$*, the graph is called* ***empty****.*

Definition 3.1.2. *Let* (V, E, η) *be a graph. The* ***degree*** *of a vertex* $v \in V$ *is* $\deg(v) := |\{e \in E \mid v \in \eta(e)\}|$*.*

Definition 3.1.3. *Let* $G = (V, E, \eta)$ *be a graph. A* ***subgraph*** *is a graph* (V', E', η') *with*

- $V' \subseteq V$*.*

- $E' \subseteq E$.
- $\eta' = \eta_{|E'}$.

For $W \subseteq V$, the subgraph $(W, \{e \in E \mid \eta(e) \subseteq W\}, \eta)$ is the ***vertex–induced subgraph of*** *W, denoted by G_W.*

Definition 3.1.4. *Let G be a graph with subgraphs (V^1, E^1, η^1) and (V^2, E^2, η^2). Their* ***intersection*** *is the subgraph $(V^1 \cap V^2, E^1 \cap E^2, \eta^1_{E_1 \cap E_2})$.*

Definition 3.1.5. *Let (V, E, η) be a graph. It is called* ***connected*** *if for each pair of vertices $v, w \in V$ there is a sequence $v_1, v_2, \ldots, v_n \in V^n$ and a sequence $e_1, e_2, \ldots, e_{n-1} \in E^{n-1}$ such that*

- *$v = v_1$ and $w = v_n$.*
- *$\eta(e_k) = \{v_k, v_{k+1}\}$ for $1 \leq k < n$.*

Definition 3.1.6. *Let $G_1 = (V_1, E_1, \eta_1)$ and $G_2 = (V_2, E_2, \eta_2)$ be two graphs. A* ***graph morphism*** *is a pair (μ_V, μ_E), where $\mu_V : V_1 \to V_2$ and $\mu_E : E_1 \to E_2$ are maps, such that $v \in \eta_1(e)$ implies $\mu_V(v) \in \eta_2(\mu_E(e))$.*

The graph morphism (μ_V, μ_E) is a ***graph isomorphism*** *if there is a graph morphism $(\hat{\mu}_V, \hat{\mu}_E) : G_2 \to G_1$ with $\hat{\mu}_V = (\mu_V)^{-1}$ and $\hat{\mu}_E = (\mu_E)^{-1}$.*

Our definition for colouring mirrors the one in [35, Subsection 5.1.1].

Definition 3.1.7. *Let (V, E, η) be a graph and M be a set. A* ***colouring*** *is a map $\kappa : V \to M$. If for every edge $e \in E$ with $\eta(e) = \{v_1, v_2\}$ we have $\kappa(v_1) \neq \kappa(v_2)$, the colouring is* ***proper****.*

3.2 Vertex–edge–graph

In this section, we work with properties of the vertex–edge–graph. Since we are mainly concerned with vertex–colourings of twisted polygonal complexes, we do not spend much time on characterisations for the other formalisations of combinatorial complexes.

Definition 3.2.1. *Let (V, E, F, η, φ) be a polygonal complex. Then, (V, E, η) is its* ***vertex–edge–graph****.*

To formalise this concept for twisted polygonal complexes, we would need to adapt our graph definition to allow loops. Since the precise definition of vertex–edge–graphs is not crucial to discuss vertex–colourings, we will not formulate this generalisation explicitly.

3.2.1 Vertex colourings

In this subsection, we define vertex colourings of a twisted polygonal complex P and relate them to twisted polygonal morphisms from P into certain simple surfaces.

Our notion of vertex–colouring coincides with the notion of proper colouring of the vertex–edge–graph ([35, Subsection 5.1.1]).

Definition 3.2.2. *Let P be a twisted polygonal complex with vertex set V, and $n \in \mathbb{N}$. A **vertex–n–colouring** is a map $c_V : V \to \{1, \dots, n\}$,*

$$c_V(\lambda_0(c)) \neq c_V(\lambda_0(\sigma_0(c)))$$

for any chamber c.

Our first result is to relate vertex–3–colourings to morphisms into a triangle. This is a well–known result. In graph–theoretical terms, it says that a graph G has an n–colouring if and only if there is a graph morphism from G to the complete graph K_n. Extensions of this result to combinatorial surfaces are presented in [12, Remark 1.6] and [10, Folgerung 2.61].

Definition 3.2.3. *The **triangle** is the twisted polygonal surface $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$, with*

$$V = \{v_1, v_2, v_3\}, \qquad E = \{e_1, e_2, e_3\}, \qquad F = \{f_1\}, \qquad C = \{c_1, \dots, c_6\},$$

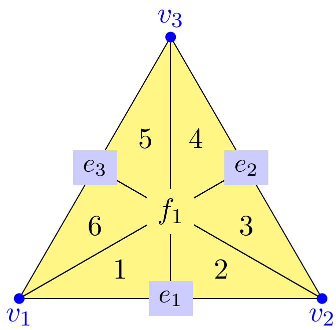
together with

$$\lambda : C \to V \times E \times F, \qquad c_k \mapsto \begin{cases} (v_1, e_1, f_1) & k = 1 \\ (v_2, e_1, f_1) & k = 2 \\ (v_2, e_2, f_1) & k = 3 \\ (v_3, e_2, f_1) & k = 4 \\ (v_3, e_3, f_1) & k = 5 \\ (v_1, e_3, f_1) & k = 6 \end{cases}$$

$$\sigma_0 = (c_1, c_2)(c_3, c_4)(c_5, c_6),$$
$$\sigma_1 = (c_1, c_6)(c_2, c_3)(c_4, c_5),$$

and $\sim$ the equivalence relation "equality".

The triangle can be illustrated as follows:



Lemma 3.2.4. *A twisted triangular complex $P = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ has a vertex–3–colouring if and only if there is a twisted polygonal morphism from P to the triangle from Definition 3.2.3.*

Proof. Let $T = (V^T, E^T, F^T, C^T, \lambda^T, \sigma_0^T, \sigma_1^T, \sim^T)$ be the triangle.

Assume there is a twisted polygonal morphism $(\mu_V, \mu_E, \mu_F, \mu_C) : P \to T$. We define a vertex–3–colouring as follows:

$$c_V : V \to \{1,2,3\} \qquad\qquad v \mapsto \begin{cases} 1 & \mu_V(v) = v_1 \\ 2 & \mu_V(v) = v_2 \\ 3 & \mu_V(v) = v_3 \end{cases}$$

It is well–defined, since any edge e of P is mapped to an edge of T, and any edge in T has two distinct vertices, the vertices incident to e also have to be distinct.

Conversely, assume c_V is a vertex–3–colouring of P. We construct a map $c_E : E \to \{\{1,2\},\{1,3\},\{2,3\}\}$ by $c_E(e) = \{c_V(v) \mid v \prec e\}$. Then, we define the twisted polygonal morphism

$$\begin{aligned}
\mu_V(v) &:= v_{c_V(v)} \\
\mu_E(e) &:= \begin{cases} e_1 & c_E(e) = \{1,2\} \\ e_2 & c_E(e) = \{2,3\} \\ e_3 & c_E(e) = \{1,3\} \end{cases} \\
\mu_F(f) &:= f_1 \\
\mu_C(c) &:= \begin{cases} c_1 & c_V(\lambda_0(c)) = 1 \wedge c_E(\lambda_1(c)) = \{1,2\} \\ c_2 & c_V(\lambda_0(c)) = 2 \wedge c_E(\lambda_1(c)) = \{1,2\} \\ c_3 & c_V(\lambda_0(c)) = 2 \wedge c_E(\lambda_1(c)) = \{2,3\} \\ c_4 & c_V(\lambda_0(c)) = 3 \wedge c_E(\lambda_1(c)) = \{2,3\} \\ c_5 & c_V(\lambda_0(c)) = 3 \wedge c_E(\lambda_1(c)) = \{1,3\} \\ c_6 & c_V(\lambda_0(c)) = 1 \wedge c_E(\lambda_1(c)) = \{1,3\}. \end{cases}
\end{aligned}$$

We have to check that this satisfies the condition of Definition 2.4.5.

1. The λ–compatibility follows from close inspection of the definitions.
2. Consider $c \in C$ and $\sigma_0(c)$. They satisfy $\lambda_1(c) = \lambda_1(\sigma_0(c))$, so they are mapped to two chambers in C^T that are in an orbit of σ_0^T.

   The same argument applies to σ_1.
3. Since $\sim^T$ is equality, we have to show that two $\sim$–equivalent chambers are mapped to the same chamber in C^T. This is clear since two $\sim$–equivalent chambers have the same values with respect to λ_0 and λ_1.
4. Since P is triangular, the non–degeneracy condition is fulfilled. □

We mention in passing that Lemma 3.2.4 can be generalised to prove the equivalence between these two statements for a twisted polygonal complex P:

- P is vertex–4–colourable.
- There is a twisted polygonal morphism from P to the tetrahedron, which was defined in Example 2.6.5.

3.3 Face–edge–graph

In this section, we work with properties of the face–edge–graph. Since we are mainly concerned with edge– and face–colourings of twisted polygonal complexes, we do not spend much time on alternative characterisations.

Definition 3.3.1. *Let (V, E, F, η, φ) be a polygonal surface with set of inner edges E_I. Then, (F, E_I, ν) with*

$$\nu : E_I \to \mathrm{Pot}_2(F) \qquad\qquad e \mapsto \{f \in F \mid e \prec f\}$$

*is its **face–edge–graph**.*

To formalise this concept for twisted polygonal complexes, we would need to adapt our graph definition to allow loops. Since the precise definition of face–edge–graphs is not crucial to discuss colourings, we will not formulate this generalisation explicitly.

It will become apparent in Section 7.3 that the possibility to colour the face–edge–graph with two colours is a very important notion.

Definition 3.3.2. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted triangular surface. A **face–2–colouring** is a map $\kappa : F \to \{1, 2\}$ such that $\kappa(\lambda_2(c_1)) \neq \kappa(\lambda_2(c_2))$, whenever $c_1 \sim c_2$ and $c_1 \neq c_2$ holds.*

3.3.1 Edge colourings

In this subsection, we work with edge colourings of twisted triangular complexes. We are interested in *Grünbaum colourings*, i. e. edge colourings of twisted triangular complexes where the three edges of a face have different colours. In [46], several of these colourings are constructed for triangulations of surfaces. To get an overview over this field, we recommend [48], which also explains the origin of the term. In [13, Version 0.5], these colourings are called *wild colourings*.

Definition 3.3.3. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted triangular complex. A map $c_E : E \to \{1, 2, 3\}$ is a **Grünbaum colouring** (or **wild colouring**) if*

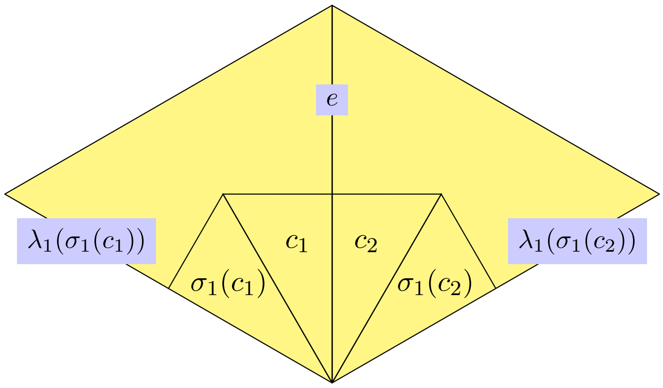
$$\{c(e) \mid e \prec f\} = \{1, 2, 3\}$$

for each face $f \in F$.

Grünbaum colourings are especially interesting for twisted triangular surfaces, since we can compare the colours of adjacent faces. There are two different ways in which adjacent faces can be coloured:

In the left image, the colours are *mirrored* across the edge. In the case on the right, they are *rotated* around the centre of the edge. This motivates the term *local symmetry* to describe these situations.

Formalising this distinction is a bit more difficult, since we have to describe “the edges adjacent to a given edge”. To do so properly, we use the chambers at one edge.

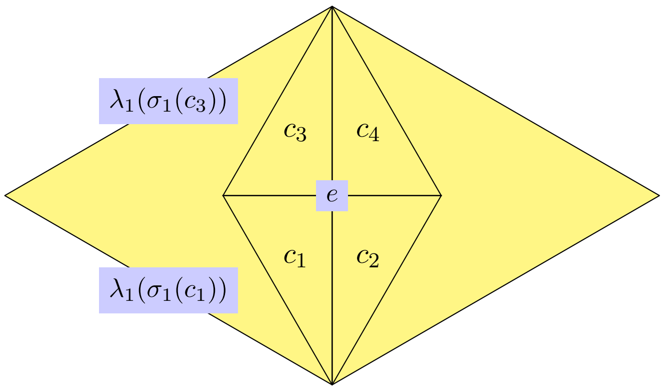


Definition 3.3.4. *Let $T = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted triangular surface and $c_E : E \to \{1, 2, 3\}$ a Grünbaum colouring. Let $e \in E$ and $\{c_1, c_2\} \subseteq C$ a $\sim$–equivalence class with $\lambda_1(c_1) = \lambda_1(c_2) = e$. The* ***local symmetry*** *of e is*

- *M if $c_E(\lambda_1(\sigma_1(c_1))) = c_E(\lambda_1(\sigma_1(c_2)))$.*
- *R if $c_E(\lambda_1(\sigma_1(c_1))) \neq c_E(\lambda_1(\sigma_1(c_2)))$.*

If all edges with colour k have the same local symmetry L_k, we call c_E an $L_1L_2L_3$– ***colouring****.*

Well–defined. We have to show that the definition of local symmetry is independent from the choice of $\sim$–equivalence classes. Let e be an edge with $\sim$–equivalence classes $\{c_1, c_2\}$ and $\{c_3, c_4\}$, such that $\sigma_0(c_1) = c_3$ and $\sigma_0(c_2) = c_4$.



Then, the edges $\lambda_1(\sigma_1(c_1))$ and $\lambda_1(\sigma_1(c_3))$ are distinct (otherwise, there would be at most two edges incident to the face $\lambda_2(c_1)$, contradicting the existence of a Grünbaum colouring for T).

Since c_E is a Grünbaum colouring, the colours of these edges are also distinct. Since this holds for $\{c_2, c_4\}$ as well, the claim follows. $\square$

In the proof of Lemma 3.2.4, we constructed an edge colouring from a given vertex colouring. We can generalise this process.

Remark 3.3.5. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted triangular complex and $c_V : V \to \{1, 2, 3\}$ a vertex–3–colouring. Then,*

$$c_E : E \to \{1,2,3\} \qquad\qquad e \mapsto \begin{cases} 1 & \{c_V(v) \mid v \prec e\} = \{2,3\} \\ 2 & \{c_V(v) \mid v \prec e\} = \{1,3\} \\ 3 & \{c_V(v) \mid v \prec e\} = \{1,2\} \end{cases}$$

is a Grünbaum colouring.

Proof. Let $f \in F$. Then, $\lambda_2^{-1}(f)$ consists of six chambers, which give at most three vertices. Since c_V is a vertex–3–colouring, Definition 3.2.2 ensures that there are three distinct vertices incident to f.

For a given $c \in \lambda_2^{-1}(f)$, we have

$$\{v \in V \mid v \prec \lambda_1(c)\} = \{\lambda_0(c), \lambda_0(\sigma_0(c))\},$$

thus all three edges incident to f have different images under c_E. $\square$

Lemma 3.3.6. *A twisted triangular surface with MMM–colouring also has a vertex–3–colouring.*

Proof. Call the surface $S = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ and the MMM–colouring c_E. Consider a vertex $v \in V$. Since S is a surface, the vertex v is either an inner or a boundary vertex. By Definition 2.4.23, there is a unique maximal strong umbrella–path around v, i. e. a sequence

$$(c_1, c_2, c_3, \dots, c_{2n})$$

with $\{c_1, c_2, \dots, c_{2n}\} = \lambda_0^{-1}(v)$, as well as

- $c_{2k} = \sigma_1(c_{2k-1})$ for $1 \le k \le n$.
- $c_{2k} \sim c_{2k+1}$ for $1 \le k < n$.

Since c_E is an MMM–colouring, $c_{2k} \sim c_{2k+1}$ implies $c_E(\lambda_1(c_{2k-1})) = c_E(\lambda_1(c_{2k+2}))$ for $1 \le k < n-1$. Since $\lambda_1(c_{2k}) = \lambda_1(c_{2k+1})$ for $1 \le k < n$, this implies

$$|\{c_E(\lambda_1(c_i)) \mid 1 \le i \le 2n\}| = 2.$$

Since c_E can only take three values, there are three two–element–subsets. We define c_V such that each vertex gets mapped to the value that does not appear in its subset.

It remains to be shown that $\mu_V(\lambda_0(c)) \ne \mu_V(\lambda_0(\sigma_0(c)))$ for any chamber $c \in C$. Let $v := \lambda_0(c)$ and $v^* := \lambda_0(\sigma_0(c))$. Then, $\lambda_0^{-1}(v)$ contains both c and $\sigma_1(c)$. Also, $\lambda_0^{-1}(v^*)$ contains $\sigma_1\sigma_0(c)$. These three chambers correspond to the three different edges of the face $\lambda_2(c)$, so their edges are mapped differently under c_E. Thus, the map μ_V takes different values on v and v^*. $\square$

It is possible to generalise this result to arbitrary Grünbaum colourings (these correspond to vertex–4–colourings), but we will not do this here. Unfortunately, this only works for spherical surfaces. The proof relies on the following theorem of `Tait`, which we cite from [35, Subsection 5.2.2].

Theorem 3.3.7 (Tait,1880,[67])**.** *A plane graph G is 4–colourable if and only if its dual G^* is 3–edge–colourable.*

Combining this result with the generalisation mentioned at the end of Subsection 3.2.1, we obtain an equivalence between Grünbaum colourings of spherical twisted triangular surfaces and twisted polygonal morphisms to the tetrahedron.

3.4 Boundary graph

In this section, we work with the boundary graph of a polygonal surface. The boundary graph is built from boundary vertices and boundary edges.

Definition 3.4.1. *Let $S = (V, E, F, \eta, \varphi)$ be a polygonal surface with boundary vertices V_B and boundary edges E_B. The graph $\partial S := (V_B, E_B, \eta_{V_B})$ is called* ***boundary graph*** *of S.*

The boundary graph has an interesting structure: It consists of several "cycles". In Subsection 3.4.1, we formalise this notion by defining cyclic graphs. We also define cyclic intervals, a generalisation of intervals in a partially ordered set to a cyclic situation. In Subsection 3.4.2, we introduce the notion of cyclic sequences and growth–control. In the final Subsection 3.4.3, we define SB–surfaces (surfaces with exactly one boundary component). These particular surfaces underlie the main construction in Chapter 8.

3.4.1 Cyclic graphs and intervals

This subsection formalises the notions of cyclic graphs and cyclic intervals. We mainly care about them because of their involvement in Chapter 8. There, we work with simplicial surfaces whose boundary is connected, and "extend" the surface along that boundary. Thus, we need a concise description of this situation.

Definition 3.4.2. *A* ***cyclic graph*** *is a finite connected graph, in which every vertex has degree 2.*

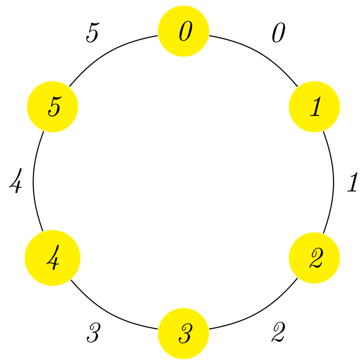
A ***cyclic interval*** *is a connected subgraph of a cyclic graph. In a cyclic interval I, vertices of degree 1 are called* ***boundary vertices****.*

Our definition of cyclic intervals is the discrete version of circular arcs, that appear for example in [21].

Example 3.4.3. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then, $(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \eta)$ with*

$$\eta : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Pot}_2(\mathbb{Z}/n\mathbb{Z}) \qquad\qquad k \mapsto \{k, k+1\}$$

is a cyclic graph. For $n = 6$, we can illustrate it like this:



The subgraph consisting of the vertices $\{0, 1, 5\}$ and the edges $\{0, 5\}$ is a cyclic interval. The subgraph with vertices $\{1, 2, 4\}$ and edges $\{1\}$ is not a cyclic interval.

Next, we analyse the shape of cyclic intervals more closely.

Lemma 3.4.4. *Let (V, E, η) be a cyclic graph and I a cyclic interval. Then, one of the following cases holds:*

1. *$I = (V, E)$.*
2. *I has no vertices and no edges.*
3. *I has exactly one vertex and no edge.*
4. *I has exactly two boundary vertices.*

With this knowledge about interval shapes, we can analyse how the intersection of two cyclic intervals looks like. To facilitate this, we prove a formal lemma about connectivity in a cyclic graph, that essentially states that there are exactly two paths from one vertex to another one.

Lemma 3.4.5. *Let (V, E, η) be a cyclic graph and $v_1, v_2 \in V$. Then, there are $E_1 \subseteq E$ and $E_2 \subseteq E$ with $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$, such that the edge set of any cyclic interval containing v_1 and v_2 has to contain E_1 or E_2.*

Proof. If $v_1 = v_2$, we can choose $E_1 := \emptyset$ and $E_2 := E$.

Otherwise, consider a cyclic interval I containing v_1 and v_2. Since I is connected, there has to be a sequence $(v_1, w_1, w_2, \ldots, w_k, v_2) \in V^{k+2}$ such that each two adjacent vertices are incident to a common edge. We can assume that the path does not contain repetitions. Since every vertex is incident to exactly two edges, knowing the edge between v_1 and w_1 determines the path. There are two options here. Each of them defines a non–repeating path from v_1 to v_2 (the graph is finite and connected). The edge sets of these two paths are E_1 and E_2. $\square$

Now, we can analyse the shape of the intersection in detail.

Lemma 3.4.6. *Let (V, E, η) be a cyclic graph and I, J cyclic intervals. If the intersection $I \cap J$ is not empty, it has one of the following forms:*

1. It is a cyclic interval contained in both I and J

2. It consists of two disjoint cyclic intervals contained in both I and J. Each of these cyclic intervals contains a boundary vertex of I and one of J.

Proof. If the intersection is empty or connected, we have one of the first cases.

Now assume that $I \cap J$ induces a disconnected graph. Then, there are vertices v_1 and v_2 in different connected components. Applying Lemma 3.4.5 to it, there are two edge sets E_1 and E_2 contained in the edge sets of I and J. If both contained the same minimal edge set, v_1 and v_2 would be connected in $I \cap J$. Therefore, we can assume $E_1 \subseteq E_I$ and $E_2 \subseteq E_J$.

Consider any other vertex $v_3 \in I \cap J$. Without loss of generality, assume v_3 lies on the path E_1. The vertex v_3 has to be connected to v_1 in J. Since v_3 lies on a path between v_1 and v_2 (and paths are unique), there has to be a path from v_3 to v_1 or v_2 in J. In particular, v_3 cannot lie in a third connected component. $\square$

Example 3.4.7. *Consider the cyclic graph from example 3.4.3 with $n = 6$. We consider the following three cyclic intervals:*

- *I has vertices $\{0, 1, 2, 3\}$ and edges $\{0, 1, 2\}$.*
- *J has vertices $\{0, 2, 3, 4, 5\}$ and edges $\{2, 3, 4, 5\}$.*
- *K has vertices $\{2, 3, 4, 5\}$ and edges $\{2, 3, 4\}$.*

Then, we have the following intersections:

- *$I \cap J$ has vertices $\{0, 2, 3\}$ and edges $\{2\}$. It is the disjoint union of the cyclic graphs with vertices $\{0\}$ and $\{2, 3\}$.*
- *$I \cap K = K$.*
- *$J \cap K$ is the cyclic interval with vertices $\{2, 3\}$ and edge $\{2\}$.*

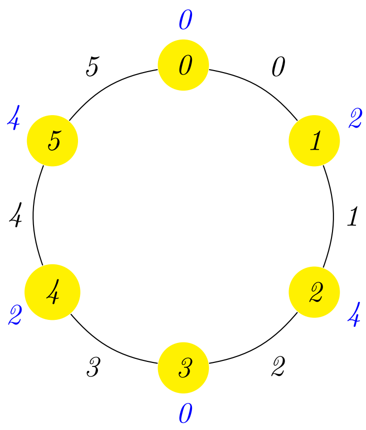
Remark 3.4.8. *Let (V, E, η) be a cyclic graph and V be the disjoint union of W_1 and W_2 such that W_1 and W_2 both induce a cyclic interval with at least 2 elements. Then each boundary vertex of W_1 is adjacent to exactly one boundary vertex of W_2.*

3.4.2 Cyclic sequences

In this subsection, we define the notion of cyclic sequences. This will be crucial in the main construction of Chapter 8.

Definition 3.4.9. *Let (V, E, η) be a cyclic graph and M a set. A **cyclic M−sequence** is a map $\varphi : V \to M$.*

Example 3.4.10. *Consider the cyclic graph from Example 3.4.3 with $n = 6$. Then, $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}, k \mapsto 2k$ is a cyclic $\mathbb{Z}/6\mathbb{Z}$−sequence. We illustrate it like this:*



We are primarily concerned with cyclic sequences over $\mathbb{N}$. Since we want to use these sequences to "control" the growth of certain extensions, we need a way to measure when a boundary becomes too "unbalanced".

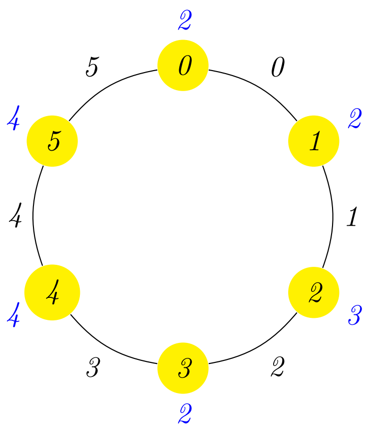
Definition 3.4.11. *Let $G = (V, E, \eta)$ be a cyclic graph, $\varphi : V \to \mathbb{N}$ a cyclic $\mathbb{N}$–sequence, and G_W a cyclic interval. The* ***defect–sum of W with respect to φ*** *is defined as*

$$d_\varphi(W) := \sum_{w \in W} (3 - \varphi(w)).$$

If $d_\varphi(W) \leq 2$, we call G_W ***defect–controlled****. The sequence φ is* ***growth–controlled*** *if every cyclic interval is defect–controlled and $d_\varphi(V) \leq 0$ holds.*

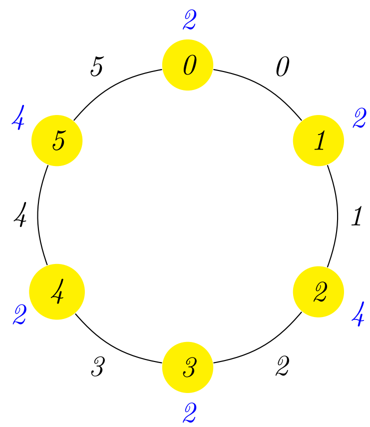
Example 3.4.12. *Consider the cyclic graph from Example 3.4.3 with $n = 6$. We represent cyclic sequences like in Example 3.4.10.*

1. *Consider the cyclic $\mathbb{N}$–sequence illustrated here:*

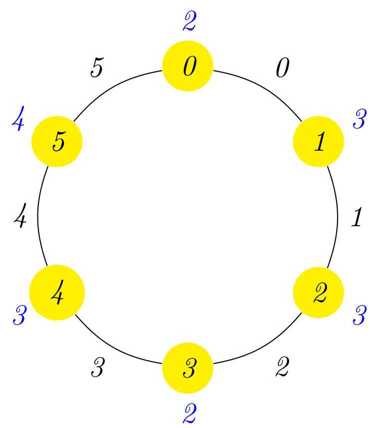
   It is not defect–controlled since the cyclic interval induced by the vertices $\{0, 1, 2, 3\}$ has defect–sum 3.

2. *Consider the cyclic $\mathbb{N}$–sequence illustrated here:*



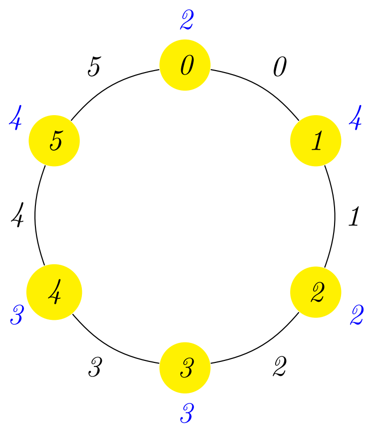
Unlike in the previous case, the cyclic interval induced by the vertices $\{0, 1, 2, 3\}$ *now has defect–sum 2. However, the cyclic interval* $\{0, 1, 2, 3, 4\}$ *has defect–sum 3, so this sequence is not growth–controlled either.*

3. *Consider the cyclic* $\mathbb{N}$*–sequence illustrated here:*



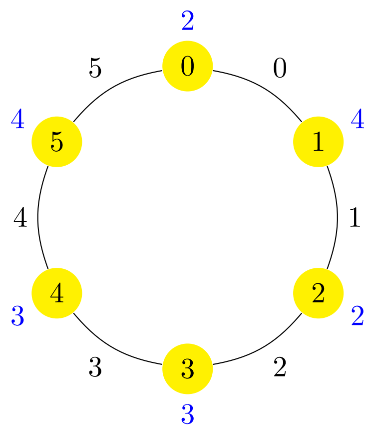
This cyclic sequence is defect–controlled. But it is not growth–controlled, since the defect–sum of all vertices is 1.

4. *Consider the cyclic* $\mathbb{N}$*–sequence illustrated here:*



This cyclic sequence is growth–controlled.

The main result of this subsection is the construction of larger growth–controlled cyclic sequences from smaller ones. For example, consider the growth–controlled cyclic $\mathbb{N}$–sequence from Example 3.4.12:



Is it possible to replace the values of the vertices in $\{1, 2\}$ by 3? Could we even add a vertex in between and set the values to $2, 3, 4$ (in order)?

In these kinds of replacements, a part of the cyclic graph stays the same, while a (usually smaller) part changes. We now formally describe how the replacement has to interact with the remaining part to stay growth–controlled. Recall the notation for the vertex–induced subgraph from Definition 3.1.3.

Lemma 3.4.13. *Let $G^1 = (V_1, E_1, \eta_1)$ and $G^2 = (V_2, E_2, \eta_2)$ be cyclic graphs with cyclic $\mathbb{N}$–sequences $\varphi_1 : V_1 \to \mathbb{N}$ and $\varphi_2 : V_2 \to \mathbb{N}$. Assume*

1. *$V_k = I_k \uplus J_k$ for $k \in \{0, 1\}$.*
2. *$G^k_{I_k}$ and $G^k_{J_k}$ are cyclic intervals.*
3. *$\rho : G^1_{J_1} \to G^2_{J_2}$ is a graph isomorphism with $\varphi_1(j) = \varphi_2(\rho(j))$ for all $j \in J_1$.*

4. φ_1 *is growth–controlled.*

Then,

1. *If* $d_{\varphi_2}(I_2) \leq d_{\varphi_1}(I_1)$*, then* $d_{\varphi_2}(V_2) \leq 0$*.*

2. *Let* S *be a cyclic interval contained in* I_2*. Denote the set of adjacent boundary vertices in* J_2 *by* $\mathcal{B}(S)$*. If there is a cyclic interval* T *contained in* I_1*, such that*
   - $\rho^{-1}(\mathcal{B}(S)) \subseteq \mathcal{B}(T)$*, where* $\mathcal{B}(T)$ *is the set of adjacent boundary vertices of* T *in* J_1*, and*
   - $d_{\varphi_1}(T) \geq d_{\varphi_2}(S)$*,*

   all cyclic intervals of V_2 *that intersect* I_2 *in* S *are defect–controlled.*

3. *Let* S *be the disjoint union of two cyclic intervals contained in* I_2 *such that each of them contains a boundary vertex of* I_2*.*

   If there is a disjoint union T *of two cyclic intervals contained in* I_1 *(each containing a boundary vertex) with* $d_{\varphi_2}(S) \leq d_{\varphi_1}(T)$*, then all cyclic intervals of* I_2 *that intersect* I_2 *in* S *are defect–controlled.*

Proof. The first claim follows from an easy calculation:

$$d_{\varphi_2}(V_2) = d_{\varphi_2}(I_2) + d_{\varphi_2}(J_2) \leq d_{\varphi_1}(I_1) + d_{\varphi_1}(J_1) = d_{\varphi_1}(V_1) \leq 0$$

For the second claim, consider a cyclic interval C with $C \cap I_2 = S$. By assumption of the lemma, there is a cyclic interval $T \subseteq I_1$. We want to show that $\rho^{-1}(C \cap J_2) \cup T$ is a cyclic interval of V_1. By Definition 3.4.2, it is sufficient to show that the induced subgraph is connected.

Let $v \in \rho^{-1}(C \cap J_2)$. Since C is a cyclic interval and $S \neq \emptyset$, there is a sequence of pairwise adjacent vertices

$$(\rho(v), v_1, v_2, \ldots, v_k, s),$$

in C, such that $s \in S$ and all $v_i \notin S$. Then, v_k is adjacent to a boundary vertex of S. By assumption on T, $\rho^{-1}(v_k)$ is also adjacent to a boundary vertex of T. Thus, there is a $t \in T$ such that

$$(v, \rho^{-1}(v_1), \rho^{-1}(v_2), \ldots, \rho^{-1}(v_k), t)$$

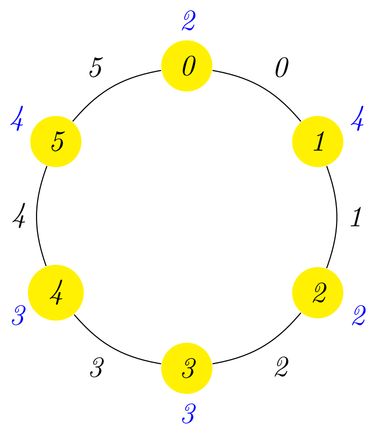
is a sequence of pairwise adjacent vertices in $\rho^{-1}(C \cap J_2) \cup T$.

Now, we have

$$\begin{aligned}
d_{\varphi_2}(C) &= d_{\varphi_2}(S) + d_{\varphi_2}(C \cap J_2) \\
&\leq d_{\varphi_1}(T) + d_{\varphi_1}(\rho^{-1}(C \cap J_2)) \\
&= d_{\varphi_1}(\rho^{-1}(C \cap J_2) \cup T) \leq 2.
\end{aligned}$$

The third situation can only appear if $J_2 \subseteq C$. Since the two disjoint cyclic intervals $T \subseteq V_1$ contain both boundary vertices of I_1, we can construct a cyclic interval just as in the second case. □

Example 3.4.14. *Let $(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \eta)$ be the cyclic graph from Example 3.4.3. Consider the following two cyclic $\mathbb{N}$–sequences, illustrated like in Example 3.4.10.*



We have seen in Example 3.4.12 that the sequence on the left (denoted φ_1) is growth–controlled. We would like to apply Lemma 3.4.13 to show that the sequence on the right (denoted φ_2) is growth–controlled.

To do so, we have to split the cyclic graph into two cyclic intervals. We choose the vertex partitions

$$I_1 = I_2 = \{1, 2\} \qquad\qquad J_1 = J_2 = \{0, 3, 4, 5\}.$$

The graph isomorphism $\rho : G^1_{J_1} \to G^2_{J_2}$ is the identity.

Now we apply the individual conclusions of Lemma 3.4.13.

1. *Since $d_{\varphi_2}(I_2) = 0 + 0 = 1 + (-1) = d_{\varphi_1}(I_1)$, we have $d_{\varphi_2}(V_2) \leq 0$.*

2. *There are three cyclic intervals contained in I_2:*
   - *Consider the cyclic interval induced by the vertex set $\{1\} \subseteq I_2$. The adjacent boundary vertices in J_2 are $\{0\}$. We choose the cyclic interval $\{1, 2\} \subseteq I_1$, since its adjacent boundary vertices are $\{0, 3\} \supseteq \{0\}$ and $d_{\varphi_1}(\{1, 2\}) = 0 = d_{\varphi_2}(\{1\})$.*
   - *Consider the cyclic interval induces by the vertex set $\{2\} \subseteq I_2$. The adjacent boundary vertices in J_2 are $\{3\}$. We choose the cyclic interval $\{2\} \subseteq I_1$ since it has the same adjacent boundary vertices and $d_{\varphi_1}(\{2\}) = 1 \geq 0 = d_{\varphi_2}(\{2\})$.*
   - *For the cyclic interval induced by I_2 itself, we also choose the cyclic interval $\{1, 2\} = I_1$.*

3. *The final case cannot happen since the boundary vertices of I_2 are adjacent.*

Thus, all cyclic intervals are defect–controlled with respect to φ_2. Thus, φ_2 is growth–controlled.

3.4.3 SB–surfaces

In the previous subsections, we defined cyclic graphs and sequences. The boundary graph of a polygonal surface (Definition 3.4.1) is the disjoint union of such graphs.

Lemma 3.4.15. *Each connected component of the boundary graph of a finite polygonal surface is cyclic.*

Proof. We check the properties of Definition 3.4.2. Since the surface is finite, the boundary graph is finite. By definition, each connected component is connected. In a polygonal surface, every boundary vertex lies in exactly one non–cyclic umbrella, that contains exactly two boundary edges. □

This has a combinatorial implication for the relation between boundary vertices and edges.

Lemma 3.4.16. *Let S be a finite combinatorial surface with boundary vertices V_B and boundary edges E_B. Then, $|V_B| = |E_B|$.*

Proof. For polygonal surfaces, this follows from Lemma 3.4.15.

For twisted polygonal surfaces and Dress–surfaces, we consider the chambers that lie at boundary edges. For each boundary edge, there are two of them. Each boundary vertex lies in a unique maximal umbrella path that contains two of these chambers. The claim follows. □

In Chapter 8, it is crucial to consider simplicial surfaces with a single boundary component. Thus, we give these surfaces a special name.

Definition 3.4.17. *A **single boundary surface** (or **SB–surface**) is a strongly connected, vertex–faithful simplicial surface S, where ∂S has exactly one boundary component.*

4 Degree–based properties

This chapter explores properties that rely on the degrees of a combinatorial surface. In Section 4.1, we introduce the concepts of degree and its complement, the defect.

Section 4.2 uses the concept of degree to define *extended surfaces*. If we consider a surface with boundary that is part of a larger surface, modelling it as an extended surface allows us to describe the immediate surroundings of the small surface.

Separate from the sections before, Section 4.3 discusses regular Dress–surfaces, whose degrees are all equal to a fixed number $d \in \mathbb{N}$. We show how these regular surfaces relate to certain subgroups of triangle groups and how to describe these subgroups.

4.1 Degrees and Defects

In this section, we introduce the concepts of degree (Definition 4.1.1) and defect (Definition 4.1.4), which complement each other.

The degree of a vertex should count the number of incident faces (differing from the graph–theory literature, where the degree counts the number of edges). Unfortunately, this description is not useful if a vertex can be incident to a face twice – in this case we would like to count with multiplicity. For that reason, we have different definitions for the different categories of combinatorial surfaces.

Definition 4.1.1. *Let (V, E, F, η, φ) be polygonal surface. For any $v \in V$ we define its* ***degree*** $\deg(v)$ *as the number of faces incident to v. Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal surface. For any $v \in V$, we define its* ***degree*** $\deg(v)$ *as $\frac{1}{2}|\lambda_0^{-1}(v)|$. Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface. For any vertex x, we define its* ***degree*** $\deg(x)$ *as $\frac{1}{2}|x|$.*

Polygonal surfaces show that the degree differs from the number of incident edges.

Remark 4.1.2. *Let (V, E, F, η, φ) be a polygonal surface and $v \in V$. If v is an inner vertex, then the number of incident edges is* $\deg(v)$*. If v is a boundary vertex, the number of incident edges is* $\deg(v) + 1$*.*

If we relate the numbers of vertices, edges, and faces to each other, the concept of degree appears naturally. In fact, these relations are a crucial property of the degree.

Lemma 4.1.3. *Let S be a triangular combinatorial surface. Let V_I be the set of inner vertices and V_B be the set of boundary vertices. Then, we have:*

$$2|E| = \sum_{v \in V_I} \deg(v) + \sum_{v \in V_B} (1 + \deg(v))$$
$$3|F| = \sum_{v \in V_I \cup V_B} \deg(v).$$

Proof. We prove this statement for the different combinatorial surfaces in turn.

- For a simplicial surface, we obtain the formulas by double–counting the incident vertex–edge–pairs (with Remark 4.1.2) and the incident vertex–face–pairs.
- For a twisted triangular surface, we count the number of chambers in several different ways:

$$C = \biguplus_{v\in V} \lambda_0^{-1}(v) = \biguplus_{e\in E} \lambda_1^{-1}(e) = \biguplus_{f\in F} \lambda_2^{-1}(f)$$

  For the vertices, Definition 4.1.1 gives

$$|C| = \sum_{v\in V} |\lambda_0^{-1}(v)| = \sum_{v\in V} 2\deg(v).$$

  For the edges, we distinguish between the set of inner edges E_I and the set of boundary edges E_B (compare Definition 2.4.10):

$$|C| = \sum_{e\in E} |\lambda_1^{-1}(e)| = \sum_{e\in E_I} |\lambda_1^{-1}(e)| + \sum_{e\in E_B} |\lambda_1^{-1}(e)| = 4|E_I| + 2|E_B|.$$

  Since all faces are triangular, Definition 2.8.9 implies

$$|C| = \sum_{f\in F} |\lambda_2^{-1}(f)| = 6|F|.$$

  We obtain the second equality by combining the equations for vertices and faces. We obtain the first equality by combining the equations for vertices and edges, if we use $|E_B| = |V_B|$ from Lemma 3.4.16.
- For a triangular Dress–surface, partition the set of chambers into different orbits. It is easy to see that we get the same results as for twisted polygonal surfaces. □

In many situations, it is more convenient to consider how much the degree of a vertex differs from 6. This relies on an interesting heuristic: Properties of combinatorial surfaces are very often related to differential–geometric properties of equilateral embeddings. Here, the defect is the discrete analogue to *curvature*. The *angular defect* from the literature (compare [66, Subsection 4.1] and [45]) is a variation of our defect.

Definition 4.1.4. *Let S be a combinatorial surface. For any inner vertex $v \in V$, we define its **defect** as* $\operatorname{def}(v) := 6 - \deg(v)$. *For any boundary vertex, the **defect** is defined as* $\operatorname{def}(v) := 3 - \deg(v)$.

*A vertex with non–zero defect is also called a **singularity**.*

We can express the `Euler`–characteristic in terms of the defects, as a discrete version of the famous `Gauss–Bonnet`–theorem (see [38, Theorem 3.10] or [64, Section 4.8] for the continuous version and [66, Subsection 4.1] or [45] for the discrete variant).

Lemma 4.1.5. *Let S be a triangular combinatorial surface and V the set of vertices. Then,*

$$6\chi = \sum_{v\in V} \operatorname{def}(v). \tag{4.1}$$

Proof. We denote the set of inner vertices by V_I and the set of boundary vertices by V_B. From Lemma 4.1.3, we obtain

$$2|E| = \sum_{v \in V_I} \deg(v) + \sum_{v \in V_B} (1 + \deg(v)) \tag{4.2}$$

and

$$3|F| = \sum_{v \in V} \deg(v). \tag{4.3}$$

We insert these equations into the formula for the `Euler`–characteristic:

$$\begin{aligned}
6\chi &= 6|V| - 6|E| + 6|F| \\
&= 6|V| - 3\sum_{v \in V_I} \deg(v) - 3\sum_{v \in V_B} (1 + \deg(v)) + 2\sum_{v \in V} \deg(v) \\
&= 6|V_I| + 6|V_B| - 3\sum_{v \in V_I} \deg(v) - 3|V_B| - 3\sum_{v \in V_B} \deg(v) + 2\sum_{v \in V} \deg(v) \\
&= 6|V_I| + 3|V_B| - \sum_{v \in V_I} \deg(v) - \sum_{v \in V_B} \deg(v) \\
&= \sum_{v \in V_I} (6 - \deg(v)) + \sum_{v \in V_B} (3 - \deg(v)) = \sum_{v \in V} \operatorname{def}(v)
\end{aligned}$$

□

4.2 Extended surfaces

Sometimes, we would like to interpret a combinatorial surface with boundary as a subsurface of a larger surface. In this section, we develop a formalism that allows us to treat a combinatorial surface "as if" it were such a subsurface, but without constructing the larger surface explicitly.

The core idea is to "pretend" that the degree of the boundary vertices is larger than it actually is. To do so, we introduce the notion of *external degree*, that measures how many "invisible" faces are incident to a boundary vertex.

Definition 4.2.1. *Let S be a combinatorial surface and V its set of vertices. An **external degree map** is a map $\widehat{\deg} : V \to \mathbb{N}$ such that*

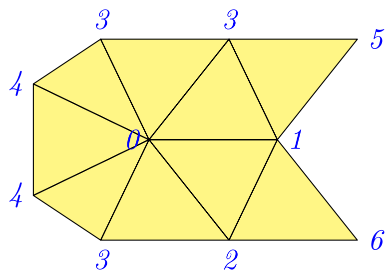
- *All inner vertices $v \in V$ satisfy $\widehat{\deg}(v) = 0$.*
- *All boundary vertices $v \in V$ satisfy $\widehat{\deg}(v) > 0$.*

*The tuple $(S, \widehat{\deg})$ is called an **extended combinatorial surface**.*

If we restrict the external degree map to the boundary vertices of a SB–surface (compare Definition 3.4.17), we obtain a cyclic $\mathbb{N}$–sequence (compare Definition 3.4.9).

Definition 4.2.2. *An **extended SB–surface** is an extended simplicial surface $(S, \widehat{\deg})$, where S is a single boundary surface. The cyclic $\mathbb{N}$–sequence $\widehat{\deg}_{|V_B}$ is called **external degree sequence**.*

Example 4.2.3. *An extended SB–surface can be visualised like this:*



Here, the blue numbers are the external degrees of the corresponding vertices.

Now, we can lift the concept *growth–controlled* from cyclic sequences to extended SB–surfaces.

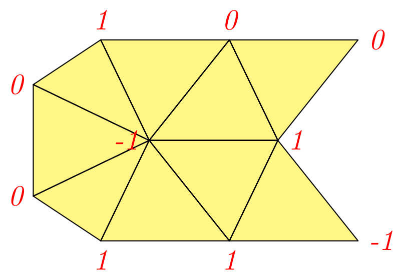
Definition 4.2.4. *An extended SB–surface is* ***growth–controlled*** *if its external degree sequence is growth–controlled and does not contain 1.*

4.2.1 Extended defects

Intuitively, the external degree of a vertex corresponds to the number of faces that are incident to that vertex but do not lie in the surface. With this interpretation, every vertex can be treated like an inner vertex. In particular, we can speak of the defect with respect to the external degree.

Definition 4.2.5. *Let S be an extended combinatorial surface. For each vertex v, we define the* ***extended defect*** *as $\widehat{\operatorname{def}}(v) := 6 - \deg(v) - \widehat{\deg}(v)$.*

Example 4.2.6. *Using the extended SB–surface from Example 4.2.3 and writing the extended defects with red labels, we obtain:*



Definition 4.1.4 called vertices with non–zero defect *singularities*. In an extended combinatorial surface, every vertex should behave like an inner vertex, so they are non–singular if their degree is 6.

Definition 4.2.7. *Let $(S, \widehat{\deg})$ be an extended combinatorial surface. A vertex v is* ***regular*** *if $\deg(v) + \widehat{\deg}(v) = 6$.*

Evaluating the Definition 4.1.4 of defect allows us to relate defect and extended defect.

Remark 4.2.8. *Let* $(S, \widehat{\deg})$ *be an extended combinatorial surface. Any inner vertex* v *satisfies* $\operatorname{def}(v) = \widehat{\operatorname{def}}(v)$*. Any boundary vertex* v *satisfies* $\widehat{\operatorname{def}}(v) = \operatorname{def}(v) + 3 - \widehat{\deg}(v)$*.*

We can reformulate the relation between defects and the `Euler`–characteristic.

Corollary 4.2.9. *Let* $(S, \widehat{\deg})$ *be an extended triangular combinatorial surface with vertex set* V*. Let* V_B *be the set of all boundary vertices. Then, we have*

$$\sum_{v \in V_B} (3 - \widehat{\deg}(v)) = \sum_{v \in V} \widehat{\operatorname{def}}(v) - 6\chi.$$

Proof. Let V_I be the set of all inner vertices. The claim follows from Lemma 4.1.5 and Remark 4.2.8:

$$\begin{aligned} 6\chi &= \sum_{v \in V} \operatorname{def}(v) \\ &= \sum_{v \in V_I} \operatorname{def}(v) + \sum_{v \in V_B} \operatorname{def}(v) \\ &= \sum_{v \in V_I} \widehat{\operatorname{def}}(v) + \sum_{v \in V_B} (\widehat{\operatorname{def}}(v) + \widehat{\deg}(v) - 3) \\ &= \sum_{v \in V} \widehat{\operatorname{def}}(v) + \sum_{v \in V_B} (\widehat{\deg}(v) - 3) \end{aligned}$$ □

4.2.2 Extended morphisms

So far, we extended combinatorial surfaces. But we can also extend morphisms in such a way. We just have to take care that the total degree (the sum of degree and external degree) does not change.

Definition 4.2.10. *Let* $(S, \widehat{\deg}_S)$ *and* $(T, \widehat{\deg}_T)$ *be two extended combinatorial surfaces. An* ***extended morphism*** *is a morphism* $\mu : S \to T$ *that fulfils*

$$\deg_S(v) + \widehat{\deg}_S(v) = \deg_T(\mu(v)) + \widehat{\deg}_T(\mu(v))$$

for all vertices v *in* S*.*

In the special situation of vertex–faithful triangular complexes, we can go even further. In this case, we can generalise the polygonal shadow and twilight morphisms from Definition 2.7.8. We also introduce a regularity concept to ensure that all "new" vertices have zero extended defect.

Definition 4.2.11. *An* ***extended polygonal twilight morphism*** *is an extended polygonal morphism that is also a polygonal shadow morphism.*

An extended polygonal twilight morphism

$$\mu : (V_S, E_S, F_S, \eta_S, \varphi_S, \widehat{\deg}_S) \to (V_T, E_T, F_T, \eta_T, \varphi_T, \widehat{\deg}_T)$$

is called ***hexagonal*** *if all vertices* $w \in V_T$ *with* $w \neq \mu_V(v)$ *for any* $v \in V_S$ *fulfil* $\deg_T(w) + \widehat{\deg}_T(w) = 6$.

Since we can interpret an extended surface as part of a larger surface, we should be able to combine two surfaces.

Lemma 4.2.12. *Let $(S, \widehat{\deg})$ with $S = (V^S, E^S, F^S, \eta^S, \varphi^S)$ be an extended simplicial surface and $T = (V^T, E^T, F^T, \eta^T, \varphi^T)$ a simplicial surface. Assume $\rho : \partial S \to \partial T$ is a graph isomorphism with $\widehat{\deg}(v) = \deg_T(\rho_V(v))$ for all boundary vertices $v \in V^S$. Then,*

$$S +_\rho T = (V^S \uplus V^T / \sim_V, E^S \uplus E^T / \sim_E, F^S \uplus F^T, \eta, \varphi),$$

with

- *$\sim_V$ is an equivalence relation on $V^S \uplus V^T$, with equivalence classes $\{v\}$ for any inner vertex $v \in V^S \uplus V^T$, and $\{v, \rho_V(v)\}$ for any boundary vertex $v \in \partial S$.*
- *$\sim_E$ is an equivalence relation on $E^S \uplus E^T$, with equivalence classes $\{e\}$ for any inner edge $e \in E^S \uplus E^T$, and $\{e, \rho_E(e)\}$ for any boundary edge $e \in \partial S$.*
- *The maps η and φ are defined as*

$$\eta : E^S \uplus E^T / \sim_E \to \mathrm{Pot}_2(V^S \uplus V^T / \sim_V)$$
$$x \mapsto \begin{cases} \{[v] \mid v \in \eta^S(e)\} & x = \{e\} \text{ for an inner edge } e \in E^S \\ \{[v] \mid v \in \eta^T(e)\} & x = \{e\} \text{ for an inner edge } e \in E^T \\ \{[v] \mid v \in \eta^S(e)\} & x = \{e, \rho_E(e)\} \text{ for a boundary edge } e \in E^S, \end{cases}$$
$$\varphi : F^S \uplus F^T \to \mathrm{Pot}_3(E^S \uplus E^T / \sim_E)$$
$$f \mapsto \begin{cases} \{[e] \mid e \in \varphi^S(f)\} & f \in F^S \\ \{[e] \mid e \in \varphi^T(f)\} & f \in F^T. \end{cases}$$

is a closed simplicial surface. Furthermore, $(S, \widehat{\deg}) \to S +_\rho T$ is an extended polygonal twilight morphism.

Proof. First, we have to show that $S +_\rho T$ is a closed simplicial surface. We start by showing the conditions of Definition 2.5.2.

1. Let $f \in F^S \uplus F^T$. Since S and T are triangular complexes, there is a sequence $(v_1, e_1, v_2, e_2, v_3, e_3)$ of incident vertices v_i and incident edges e_i. Then, $([v_1]_{\sim_V}, [e_1]_{\sim_E}, [v_2]_{\sim_V}, [e_2]_{\sim_E}, [v_3]_{\sim_V}, [e_3]_{\sim_E})$ is the sequence for f in $S +_\rho T$.
2. Let $[v]$ be a vertex in $V^S \uplus V^T / \sim_V$, then there is an element $w \in [v]$ with $w \in V^S$ or $w \in V^T$. In either case, there is an edge $e \in E^S$ (or $e \in E^T$) with $v \prec e$. By definition of η, we have $[v] \in \eta([e])$.

   A similar argument shows that every edge of $S +_\rho T$ is incident to a face.

Next, we show that all edges of $S +_\rho T$ are inner edges. This is clear for all edges $[e] \in E^S \uplus E^T / \sim_E$ with $|[e]| = 1$. Consider an edge $\{e^S, e^T\}$ with $e_S \in E^S$ and $e^T \in E^T$. By definition, e^S is a boundary edge. Since $e^T = \rho_E(e^S)$, the edge e^T is a boundary edge as well. Thus, they are both incident to exactly one face. Since $\{e^S, e^T\}$ is incident to all faces that are incident to at least one of their elements, this edge is incident to exactly two faces.

Finally, we have to show that there are no ramified vertices. Again, this is easy for all inner vertices of S and T, since all incident edges are inner edges. Consider a vertex $\{v^S, v^T\}$ with $v^S \in V^S$ and $v^T \in V^T$. Then, both are boundary vertices, so there are maximal non–closed umbrellas

$$(e_0^S, f_1^S, e_1^S, f_2^S, \dots, f_n^S, e_n^S) \qquad\qquad (e_0^T, f_1^T, e_1^T, f_2^T, \dots, f_m^T, e_m^T).$$

Since $\rho : \partial S \to \partial T$ is a graph isomorphism, we have $\{\rho_E(e_0^S), \rho_E(e_n^S)\} = \{e_0^T, e_m^T\}$. Without loss of generality, say $\rho_E(e_0^S) = e_0^T$ and $\rho_E(e_n^S) = e_m^T$. Then,

$$([e_0^S], f_1^S, [e_1^S], \dots, f_n^S, [e_n^S], f_m^T, [e_{m-1}^T], \dots, [e_1^T], f_1^T)$$

is a maximal closed umbrella around $\{v^S, v^T\}$.

To complete the proof, we have to show that $(S, \widehat{\deg}) \to S +_\rho T$ is an extended polygonal twilight morphism. But this follows from $\widehat{\deg}(v) = \deg_T(\rho_V(v))$. □

4.3 Regular Dress–surfaces

In this section, we analyse the particular case of regular Dress–surfaces, i. e. closed Dress–surfaces with constant degree. We want to characterise them group–theoretically.

The theoretical background (coset actions) is covered in Subsection 4.3.1. In Subsection 4.3.2, we construct a correspondence between regular Dress–surfaces and certain subgroups of triangle groups. Finally, in Subsection 4.3.3, we will extend this correspondence to a connection between the subgroup inclusion and Dress covering morphisms.

Since the degree of the vertices is relevant, we choose a name that emphasises it.

Definition 4.3.1. *Let $S = (C, \sigma_0, \sigma_1, \sigma_2)$ be a triangular, closed Dress–surface such that $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ acts transitively on C. If all vertices have the same degree d, then S is called a **degree–d Dress–surface**.*

For degree–d Dress–surfaces, the orbits of $\langle \sigma_1, \sigma_2 \rangle$ all have the same size.

Corollary 4.3.2. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a degree–d Dress–surface. Then, $\langle \sigma_0, \sigma_1 \rangle$ only has orbits of size 6 on C, $\langle \sigma_0, \sigma_2 \rangle$ only has orbits of size 4, and $\langle \sigma_1, \sigma_2 \rangle$ only has orbits of size $2d$.*

Proof. We have $\langle \sigma_0, \sigma_1, \sigma_2 \rangle \le \mathrm{Sym}(C)$ and all of them are involutions. We consider the different subgroups in turn.

- Consider the subgroup $\langle \sigma_0, \sigma_1 \rangle$. By Definition 4.3.1, the Dress–surface is triangular. Definition 2.8.9 then tells us that all faces contain 6 elements. Since faces are just the orbits of $\langle \sigma_0, \sigma_1 \rangle$ on C (Definition 2.6.3), the claim follows.

- Consider the subgroup $\langle \sigma_0, \sigma_2 \rangle$. By Definition 4.3.1, the Dress–surface is closed, so all edges are inner edges (Definition 2.8.6). By Definition 2.6.7, this means that all edges contain 4 elements. Since edges are the orbits of $\langle \sigma_0, \sigma_2 \rangle$ on C (Definition 2.6.3), the claim follows.

- Consider the subgroup $\langle \sigma_1, \sigma_2 \rangle$, whose orbits are the vertices by Definition 2.6.3. By Definition 4.3.1, all vertex degrees are d, so all vertices contain $2d$ elements (by Definition 4.1.1). □

4.3.1 Coset actions

In this subsection, we recall some basic facts about coset actions. In general, we will act from the left. The material here is covered in more detail in the group theory literature, including [7], [54], and [2].

We start by setting down the notation of group actions. There are two primary ways to describe group actions and we make use of both of them.

Definition 4.3.3. *Let G be a group and M be a set. A **(left) action** of G on M is a map $G \times M \to M, (g, m) \mapsto g.m$ satisfying*

1. *$1.m = m$ for all $m \in M$ and*
2. *$g.(h.m) = (gh).m$ for all $m \in M$ and $g, h \in G$.*

*The **action homomorphism** of this action is the group homomorphism*

$$G \to \mathrm{Sym}(M) \qquad\qquad g \mapsto (m \mapsto g.m).$$

*For any group homomorphism $\gamma : G \to \mathrm{Sym}(M)$, the **action via** γ denotes the action*

$$G \times M \to M \qquad\qquad (g, m) \to \gamma(g)(m).$$

The basic equality concept for actions is *equivariance*.

Definition 4.3.4. *Let G be a group acting on the sets M and N. The actions are called **equivariant** if there is a bijection $\varphi : M \to N$ with $\varphi(g.m) = g.\varphi(m)$ for all $g \in G$ and $m \in M$.*

It is well–known that every transitive group action is equivariant to a coset action (e. g. [7, (5.8)], [54, Theorem 6.3], and [2, Proposition 9.9]):

Theorem 4.3.5. *Let G be a group acting transitively on the set M. For any $m \in M$, the action of G on M is equivariant to the action of G on the cosets G/U, with $U := \mathrm{Stab}_G(m)$, by the bijection*

$$G/U \to M \qquad\qquad gU \mapsto g.m$$

Two coset actions G/U and G/V are equivariant if and only if U and V are conjugate. If m is replaced by $h.m$, the subgroup U is replaced by hUh^{-1}.

The subgroup U in Theorem 4.3.5 depends on a choice of $m \in M$. For convenience, we state the equivariance connecting different choices explicitly.

Remark 4.3.6. *Let G be a group and $U \leq G$ a subgroup. For $h \in G$ define $V := hUh^{-1}$. Then, the following map is an equivariance of the coset actions:*

$$\varphi : G/U \to G/V \qquad\qquad tU \mapsto th^{-1}V$$

Proof. φ is clearly bijective. For any $g \in G$ we have

$$\varphi(g.tU) = \varphi(gtU) = gth^{-1}V = g.\varphi(tU)$$ □

Next, we describe the equivariances from G/U to itself.

Lemma 4.3.7. *Let G be a group and $U \leq G$ a subgroup. $N_G(U)$ acts on G/U via $(n, gU) \mapsto gn^{-1}U$. The action homomorphism of this action induces an isomorphism between $N_G(U)$ and the group of equivariances $G/U \to G/U$.*

Proof. Let $\varphi : G/U \to G/U$ be an equivariance. Then, there is an $n \in G$ such that $\varphi(U) = nU$. From there, we conclude

$$\varphi(gU) = g\varphi(U) = gnU.$$

It remains to check for which n this map is well–defined.

Let $u \in U$, then $\varphi(gU) = \varphi(guU)$ for all $g \in G$. Equivalently, $gnU = gunU$, so $n^{-1}un \in U$. Since this holds for all $u \in U$, we conclude $n \in N_G(U)$.

Consider the second claim. Let $n, m \in N_G(U)$, then

$$m.(n.gU) = m.(gn^{-1}U) = gn^{-1}m^{-1}U = g(mn)^{-1}U = mn.gU,$$

so $(n, gU) \mapsto gn^{-1}U$ defines an action. □

As a small application of the normaliser action, we prove an invariant for its action on coset tuples.

Lemma 4.3.8. *Let G be a group and $U \leq G$ a subgroup. The left action of $N_G(U)$ on $G/U \times G/U$ (from Lemma 4.3.7) has the orbit–distinguishing invariant*

$$(aU, bU) \mapsto (aN_G(U), aUb^{-1}).$$

Proof. Since $U \leq N_G(U)$, the first component is invariant. For the second component:

$$n.(aU, bU) = (an^{-1}U, bn^{-1}U) \mapsto (an^{-1})U(bn^{-1})^{-1} = an^{-1}Unb^{-1} = aUb^{-1},$$

since $n \in N_G(U)$. This shows the invariance.

Finally we have to show that this invariant distinguishes between the orbits of the normaliser–action. Assume that we have two pairs of cosets (aU, bU) and (cU, dU) with the same invariant. In particular, we know that $aN_G(U) = cN_G(U)$. Therefore, there exists an $n \in N_G(U)$ such that $an = c$. Applying n to (aU, bU) gives (cU, bnU). The second component of the invariant gives $cU(bn)^{-1} = cUd^{-1}$, or $Un^{-1}b^{-1} = Ud^{-1}$. This shows that the invariant is orbit–distinguishing. □

We can apply Lemma 4.3.7 to describe the automorphism group of a connected Dress–surface $(C, \sigma_0, \sigma_1, \sigma_2)$: In Definition 5.2.15, we see that the connectivity of the Dress–surface is equivalent to the transitivity of $G := \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ on C. Thus, Theorem 4.3.5 gives an equivariance between C and G/U with $U := \mathrm{Stab}_G(c)$ for any $c \in C$.

Since automorphisms of a Dress–surface are bijections that are compatible with the group action (compare Definition 2.6.8), they correspond to equivariances from G/U to itself. By Lemma 4.3.7, they correspond to elements of $N_G(U)$.

The kernel of the corresponding action homomorphism $N_G(U) \to \mathrm{Sym}(G/U)$ is U. Thus, the automorphism group of the Dress–surface is isomorphic to $N_G(U)/U$.

This result actually holds in much greater generality, see [42, page 238].

4.3.2 Group–theoretic characterisation

In this subsection, we construct the correspondence between connected degree–d Dress–surfaces and certain subgroups of triangle groups.

Triangle groups are well–known in the literature, compare e. g. [34, Section 6.2.8] for a detailed introduction. For our purposes, we only need the triangle groups classically referred to as $T(2, 3, d)$, so we define them as follows:

Definition 4.3.9. *For $d \in \mathbb{N}$, the* ***triangle group*** *is the finitely presented group*

$$T_d := \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^2, (bc)^d \rangle. \tag{4.4}$$

It is interesting to note that triangle groups appear naturally as automorphism groups of certain regular surfaces. From [51, Chapter II], we obtain these correspondences (and refer to it for further detail on this tangent):

$d \in \{3, 4, 5\}$	Sphere (Tetrahedron, Octahedron, Icosahedron)
$d = 6$	Euclidean plane
$d \geq 7$	Hyperbolic plane

There is a strong connection between triangle groups and degree–d Dress–surfaces, since a triangle groups acts on the set of flags. This action was used in [23] to classify certain regular surfaces (in the formalism of combinatorial maps).

Remark 4.3.10. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a connected degree–d Dress–surface. Then, T_d acts transitively on C (from the left) via the action homomorphism defined by*

$$T_d \to \mathrm{Sym}(C) \qquad a \mapsto \sigma_0 \qquad b \mapsto \sigma_1 \qquad c \mapsto \sigma_2.$$

Proof. By Definition 4.3.3, it is sufficient to show that the given homomorphism is well–defined. Let $G = \langle \alpha, \beta, \gamma \rangle$ be the free group with three generators and consider the group homomorphism defined by

$$\varphi : G \to \mathrm{Sym}(C) \qquad \alpha \mapsto \sigma_0 \qquad \beta \mapsto \sigma_1 \qquad \gamma \mapsto \sigma_2.$$

By Definition 2.6.1, σ_0, σ_1, and σ_2 are involutions. Thus, $\alpha^2, \beta^2, \gamma^2 \in \ker(\varphi)$. Definition 2.6.1 also implies $(\sigma_0\sigma_2)^2 = id_C$, so $(\alpha\gamma)^2 \in \ker(\varphi)$.

By Corollary 4.3.2, the orbits of $\langle \sigma_0, \sigma_1 \rangle$ have size 6. Since neither σ_0 nor σ_1 have fixed points (Definition 2.6.1), the orbit of the chamber $c \in C$ is

$$\{c, \sigma_1(c), \sigma_0\sigma_1(c), \sigma_1\sigma_0\sigma_1(c), (\sigma_0\sigma_1)^2(c), \sigma_1(\sigma_0\sigma_1)^2(c)\}.$$

In particular, $(\sigma_0\sigma_1)^3(c) = c$ for all chambers $c \in C$. Since $(\sigma_0\sigma_1)^3 \in \mathrm{Sym}(C)$, this implies $(\sigma_0\sigma_1)^3 = id_C$. Thus, $(\alpha\beta)^3 \in \ker(\varphi)$.

Finally, σ_2 cannot have fixed points (otherwise $\langle \sigma_0, \sigma_2 \rangle$ would have an orbit of size 2), so we can apply the same argument as above to obtain $(\sigma_1\sigma_2)^d = id_C$. This implies $(\beta\gamma)^d \in \ker(\varphi)$.

By the homomorphism theorem, the group homomorphism described in Remark 4.3.10 is well–defined. ☐

Theorem 4.3.5 allows us to rewrite the action from Remark 4.3.10 as a coset action.

Corollary 4.3.11. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a degree–d Dress–surface, $c \in C$, and $U := \mathrm{Stab}_{T_d}(c)$. Then, $T_d/U \to C$, $gU \mapsto g.c$ is an equivariance between the action from Remark 4.3.10 and the left action of T_d on the cosets T_d/U.*

Every class of isomorphic degree–d Dress–surfaces defines a conjugacy class of subgroups. We want to classify all subgroups that correspond to a degree–d Dress–surface.

Definition 4.3.12. *A subgroup $U \leq T_d$ is called* ***surface subgroup*** *if the coset action of T_d on T_d/U is equivariant to the action of T_d on the chambers of a degree–d Dress–surface.*

Lemma 4.3.13. *The subgroup $U \leq T_d$ is a surface subgroup if and only if the groups $\langle a, b \rangle$, $\langle a, c \rangle$, and $\langle b, c \rangle$ act regularly on their coset orbits in T_d/U.*

In this case $(T_d/U, a, b, c)$ is a degree–d–surface.

Proof. The subgroups $\langle a, b \rangle$, $\langle a, c \rangle$, and $\langle b, c \rangle$ are dihedral groups of orders 6, 4, and $2d$, respectively. For example $\langle a, b \rangle \cong \langle x, y \mid x^2, y^2, (xy)^3 \rangle \cong D_6$.

Suppose first that $U \leq T_d$ is a surface subgroup, i. e. there is a degree–d Dress–surface $(C, \sigma_0, \sigma_1, \sigma_2)$ such that the actions of T_d on C and T_d/U are equivariant. Since $\langle a, b \rangle$ acts on T_d/U as $\langle \sigma_0, \sigma_1 \rangle$ acts on C (by Remark 4.3.10), and $\langle \sigma_0, \sigma_1 \rangle$ acts regularly on each of its orbits, $\langle a, b \rangle$ also acts regularly. Similar arguments apply to $\langle a, c \rangle$ and $\langle b, c \rangle$.

Conversely, $(T_d/U, a, b, c)$ is a degree–d–surface (compare Definition 2.6.1):

- Since $\langle a, b, c \rangle = T_d$, it acts transitively on T_d/U.
- By the definition of T_d, the elements a, b, and c are involutions.

  If a fixed a coset gU, the orbits of gU under $\langle a, b \rangle$ would have size at most 2. Then, $\langle a, b \rangle$ could not be acting regularly on this orbit. Therefore, a cannot fix a coset. The same argument applies to b and c as well.

- By the definition of T_d, we have $(ab)^3 = 1$. If there was a coset gU such that $(ab)^k.gU = gU$ for some $0 < k < 3$, the group $\langle a, b\rangle$ would not be acting regularly on the orbit $\langle a, b\rangle.gU$. Therefore, ab has only 3–cycles.

  The same argument applies to ac and bc. □

We can characterise surface subgroups without reference to coset actions.

Lemma 4.3.14. *A subgroup $U \leq T_d$ is a surface subgroup if and only if $gUg^{-1} \cap X = \{1\}$ for all $g \in T_d$ and all $X \in \{\langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle\}$.*

Proof. We apply the characterisation from Lemma 4.3.13.

The group $\langle a, b\rangle$ acts non–regularly if any $1 \neq x \in \langle a, b\rangle$ fixes a coset gU, i. e. $x.gU = gU$. This is equivalent to $g^{-1}xg \in U$, or $x \in gUg^{-1}$. Therefore, the action is non–regular if and only if $gUg^{-1} \cap \langle a, b\rangle \neq \{1\}$. The arguments for $\langle a, c\rangle$ and $\langle b, c\rangle$ are similar. □

4.3.3 Coverings

In Corollary 4.3.11, we obtained a correspondence between degree–d Dress–surfaces and surface subgroups. In this subsection, we show a deeper underlying correspondence: It is possible to describe coverings with it. We focus on the correspondence between subgroup inclusion and Dress covering morphisms.

We do not present the full discrete covering theory in this thesis, but it can be found (for simplicial complexes) in the excellent reference [58].

Lemma 4.3.15. *Let $U \leq V \leq T_d$ be two surface subgroups. Then, there is a Dress covering morphism $T_d/U \to T_d/V, \quad tU \mapsto tV$.*

Proof. Call the map φ and the Dress–surfaces $(T_d/U, a, b, c)$ and $(T_d/V, a, b, c)$. First, we have to show that φ is a Dress morphism according to Definition 2.6.8. Since

$$\varphi(x.tU) = xtV = x.\varphi(tU)$$

holds for all $x \in \{a, b, c\}$, the map is compatible with the involutions.

The restriction $\varphi : \langle a, b\rangle.tU \to \langle a, b\rangle.tV$ is bijective since the action of the subgroup $\langle a, b\rangle$ is regular by Lemma 4.3.13.

Since Lemma 4.3.13 also applies to the subgroups $\langle a, c\rangle$ and $\langle b, c\rangle$, the Dress morphism φ is even a Dress covering morphism (compare Definition 2.6.10). □

Remark 4.3.16. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ and $(D, \tau_0, \tau_1, \tau_2)$ be two degree–d Dress–surfaces and $\varphi : C \to D$ a Dress covering morphism. For each $c \in C$, we have*

$$\mathrm{Stab}_{T_d}(c) \leq \mathrm{Stab}_{T_d}(\varphi(c)).$$

Proof. If $x \in \mathrm{Stab}_{T_d}(c)$, the chambers c and $x.c$ are mapped to the same chamber $\varphi(c)$ in D. In particular, $x \in \mathrm{Stab}_{T_d}(\varphi(c))$. □

This gives a correspondence between surface subgroups of T_d and degree–d–surfaces. It sends inclusions to coverings and maps conjugated subgroups to isomorphic surfaces.

5 Topological concepts

In this chapter, we discuss several aspects of combinatorial surfaces that are related to topology. In Section 5.1, we construct the topological realisation of a twisted polygonal complex. In Section 5.2, we explore different concepts of connectivity for combinatorial surfaces, and in Section 5.3, we explore orientability–concepts.

5.1 Topological realisation of twisted polygonal complexes

In this section, we construct the topological realisation of a twisted polygonal complex. Since we need some topological definitions, these are covered in Subsection 5.1.1. The actual construction is presented in Subsection 5.1.2.

5.1.1 Basic topological definitions

In this subsection, we collect several elementary topological statements that are necessary to construct the topological realisation of a twisted polygonal complex in Subsection 5.1.2. All of them can be found in most introductory texts about elementary topology, including [43], [29], [6], and [32]. We mostly follow the presentation of [43].

The most basic definition in topology is that of a topological space. Intuitively, this definition formalises the concept of "closeness", but without having a distance.

Definition 5.1.1. *A **topological space** is a pair $(X, \mathcal{O})$ consisting of a set X and a set $\mathcal{O} \subseteq \mathrm{Pot}(X)$ (the elements of $\mathcal{O}$ are called **open sets**) such that the following axioms hold:*

- *Any union of open sets is an open set.*
- *The intersection of two open sets is an open set.*
- *$\emptyset$ and X are open sets.*

Topological spaces are connected by morphisms. These morphisms are maps that preserve the topological structure. Such maps are called *continuous*.

Definition 5.1.2. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A map $\psi : X \to Y$ is **continuous** if $\psi^{-1}(B) \in \mathcal{O}_X$ for all $B \in \mathcal{O}_Y$.*

*If ψ is a continuous bijection and ψ^{-1} is continuous, ψ is called a **homeomorphism**.*

Up to this point, we have not constructed any topological space. Since we want to construct one, we need some examples.

Example 5.1.3. *$(\mathbb{R}^n, \mathcal{O})$ is a topological space, where $A \in \mathcal{O}$ if for every $x \in A$, there is an $\varepsilon > 0$ such that*

$$\left\{ y \in \mathbb{R}^n \;\middle|\; \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} < \varepsilon \right\} \subseteq A.$$

Since our goal is to construct the topological realisation of a twisted polygonal complex in Subsection 5.1.2, we are especially interested in constructing new topological spaces from already known spaces (like $\mathbb{R}^n$ from Example 5.1.3). One of these constructions is combining two topological spaces disjointly.

Remark 5.1.4. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. Then the **disjoint union** $(X \uplus Y, \{A \uplus B \mid A \in \mathcal{O}_X, B \in \mathcal{O}_Y\})$ is a topological space.*

Obviously, the disjoint union of copies of $\mathbb{R}^n$ is not really useful yet, so we need to be able to construct some more interesting spaces. Fortunately, $\mathbb{R}^n$ is large enough to contain many interesting subsets. Each of them can be interpreted as a topological space as well.

Definition 5.1.5. *Let $(X, \mathcal{O})$ be a topological space and $S \subseteq X$. Then $(S, \{A \cap S \mid A \in \mathcal{O}\})$ is a topological space, called **subspace topology** or **induced topology**.*

With Definition 5.1.5 and Example 5.1.3, we can construct topological polygons. To do so, we employ the well–known identification $\mathbb{C} \cong \mathbb{R}^2$ that does not change the open sets of $\mathbb{R}^2$. This allows us to describe the boundary points of a regular polygon in a concise way.

Definition 5.1.6. *Let $n \geq 3$. The **standard** n**–gon** is the topological space induced by the convex hull of $\{e^{\frac{2\pi i}{n}k} \mid 1 \leq k \leq n\} \subseteq \mathbb{C} \cong \mathbb{R}^2$.*

Since a twisted polygonal complex consists of several polygons, we have to combine them. This is done by a "gluing" process, that is formalised by the concept of *quotient topology*. To describe this topology, we have to fix some notation about equivalence relations, which we copy from [43, Chapter III].

Notation 5.1.7. *If X is a set and $\sim$ an equivalence relation on X, then:*

- *$X/\sim$ denotes the set of equivalence classes.*
- *$[x] \in X/\sim$ denotes the equivalence class of $x \in X$.*
- *$\pi : X \to X/\sim, \quad x \mapsto [x]$ denotes the canonical projection.*

Now, we can describe the quotient topology. It is the natural choice for a topology if we start from a topological space with equivalence relation and transfer the topology to the space of equivalence classes.

Definition 5.1.8. *Let* $(X, \mathcal{O})$ *a topological space and* $\sim$ *an equivalence relation on* X*. The topological space* $(X/\sim, \mathcal{O}/\sim)$ *is called* ***quotient topology****, where*

$$A \in \mathcal{O}/\sim \quad \Leftrightarrow \quad \pi^{-1}(A) \in \mathcal{O}.$$

Since we do not just identify single points, but the edges of polygons, we use the concepts of *paths* to formulate the identifications more concisely.

Definition 5.1.9. *Let* $(X, \mathcal{O})$ *be a topological space. A* ***path*** *is a continuous map* $[0, 1] \to X$*. The* ***path space of*** X *is the set of all paths and denoted by* $X^{[0,1]}$*.*

To improve our understanding of *connectivity* in Section 5.2, we define connectivity for topological spaces.

Definition 5.1.10. *A topological space* $(X, \mathcal{O})$ *is* ***path–connected*** *if, for every two points* $x_1, x_2 \in X$*, there is a path* $p : [0, 1] \to X$ *with* $p(0) = x_1$ *and* $p(1) = x_2$*.*

We also introduce `Jordan`–curves for later use.

Definition 5.1.11. *A* ***`Jordan`–curve*** *is a path* $p : [0, 1] \to \mathbb{R}^2$ *such that* $p(0) = p(1)$ *and* $p(a) \neq p(b)$ *for all* $0 \leq a < b < 1$*.*

At this point, we have introduces enough topological concepts to start the construction of the topological realisation for twisted polygonal complexes in Subsection 5.1.2.

5.1.2 Construction

In this subsection, we construct the topological realisation of a twisted polygonal complex. We start with a discussion about the options available in the literature to realise simplicial complexes topologically. There exist several options, which are described in more detail in [47, Section 2.2].

- [62] defines a functor from the category Simp to the category of topological spaces (with continuous functions) via a combinatorial definition of the barycentric coordinate system. In [31, Chapter II], this construction is taken as the definition of a simplicial complex.

  The same construction is described in [47] as *convex combinations*.

- For finite simplicial complexes, [47] defines a topological realisation by giving all vertices explicit coordinates in a sufficiently high–dimensional vector space.

- For (not necessarily finite) simplicial complexes, [47] describes a "gluing" approach to construct a topological realisation.

Not all of these approaches can be generalised to twisted polygonal complexes. Both the combinatorial construction and giving explicit coordinates assume that a simplex is defined uniquely by its vertices. This is an assumption that is not fulfilled for twisted polygonal complexes in general. Kozlov hints at this problem when he talks about *polyhedral complexes* and *generalized simplicial complexes*, which he defines by a gluing process – since that is the only remaining approach.

We now give an explicit construction of the topological realisation of a twisted polygonal complex. We proceed as follows:

1. Define a (topological) polygon for each face.

2. Glue these polygons together along their boundaries, according to the incidence information from vertices and edges.

We start by defining the polygons for each face of the twisted polygonal complex $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$. The basis for these are the topological polygons from Definition 5.1.6.

However, we have to go one step further: We want to keep the connection between the combinatorial properties of a face and the topological polygon. To do so, we map each chamber c of the given face to a path (compare Definition 5.1.9) that starts at the point corresponding to the vertex $\lambda_0(c)$ and goes straight along the boundary corresponding to the edge $\lambda_1(c)$, until it reaches the midpoint.

Definition 5.1.12. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex. Let $(c_1, c_2, \dots, c_{2n})$ be a strong polygon path of the face $f \in F$. A* ***twisted topological polygon of*** *f is a topological space P_f, together with a map $\pi_f : \{c_1, \dots, c_{2n}\} \to P_f^{[0,1]}$, such that*

1. *There is a homeomorphism φ from the standard n–gon to P_f.*

2. *The map π_f fulfils the following property:*

$$\pi_f(c_m) = \varphi \circ \begin{cases} t \mapsto (1-t)e^{\frac{2\pi i}{n}k} + t\frac{e^{\frac{2\pi i}{n}k} + e^{\frac{2\pi i}{n}(k+1)}}{2} & m = 2k+1 \\ t \mapsto (1-t)e^{\frac{2\pi i}{n}k} + t\frac{e^{\frac{2\pi i}{n}k} + e^{\frac{2\pi i}{n}(k-1)}}{2} & m = 2k \end{cases}$$

Well–defined. We have to show that the twisted topological polygon of f is independent from the chosen strong polygon path. Let $(c_1, \dots, c_{2n})$ and $(d_1, \dots, d_{2n})$ be two such paths. Then, by Remark 2.4.16, we can obtain one from the other by cyclic permutation and reflection.

Thus, it is sufficient to show that the definition of twisted topological polygons is invariant under these operations. The cyclic permutation maps the strong polygon path $(c_1, \dots, c_{2n})$ to the strong polygon path $(c_3, c_4, \dots, c_{2n}, c_1, c_2)$. Let (P_1, π_1) be the twisted topological polygon defined from the first strong polygon path, and (P_2, π_2) the one defined from the second one.

We have two homeomorphism $\varphi_1 : Q \to P_1$ and $\varphi_2 : Q \to P_2$, where Q is the standard n–gon. Then, we define the homeomorphism

$$\tau : P_1 \to P_2 \qquad\qquad x \mapsto \varphi_2(\varphi_1^{-1}(x) \cdot \mathrm{e}^{-\frac{2\pi}{n}}).$$

This gives

$$
\begin{aligned}
\pi_2(c_m) &= \varphi_2 \circ \begin{cases} t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}(k-1)} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}(k-1)}+\mathrm{e}^{\frac{2\pi i}{n}k}}{2} & m = 2k+1 \\ t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}(k-1)} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}(k-1)}+\mathrm{e}^{\frac{2\pi i}{n}(k-2)}}{2} & m = 2k \end{cases} \\
&= \varphi_2 \circ (y \mapsto \mathrm{e}^{-\frac{2\pi}{n}} y) \circ \begin{cases} t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}k} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}k}+\mathrm{e}^{\frac{2\pi i}{n}(k+1)}}{2} & m = 2k+1 \\ t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}k} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}k}+\mathrm{e}^{\frac{2\pi i}{n}(k-1)}}{2} & m = 2k \end{cases} \\
&= \varphi_2 \circ (y \mapsto \mathrm{e}^{-\frac{2\pi}{n}} y) \circ \varphi_1^{-1} \circ \pi_1(c_m) \\
&= \tau \circ \pi_1(c_m)
\end{aligned}
$$

The reflection maps $(c_1, \ldots, c_{2n})$ to $(c_{2n}, c_{2n-1}, \ldots, c_2, c_1)$. With the same notation as before, the homeomorphism τ is defined by

$$
\tau : P_1 \to P_2 \qquad\qquad x \mapsto \varphi_2(\overline{\varphi_1^{-1}(x)}),
$$

where $\overline{y}$ is the complex conjugate of $y \in \mathbb{C}$. Then, with $\mathrm{e}^{2\pi i} = 1$, we get

$$
\begin{aligned}
\pi_2(c_m) &= \varphi_2 \circ \begin{cases} t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}(n-k)} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}(n-k)}+\mathrm{e}^{\frac{2\pi i}{n}(n-k-1)}}{2} & m = 2k+1 \\ t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}(n-k)} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}(n-k)}+\mathrm{e}^{\frac{2\pi i}{n}(n-k+1)}}{2} & m = 2k \end{cases} \\
&= \tau \circ \varphi_1 \circ \begin{cases} t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}(k-n)} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}(k-n)}+\mathrm{e}^{\frac{2\pi i}{n}(k+1-n)}}{2} & m = 2k+1 \\ t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}(k-n)} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}(k-n)}+\mathrm{e}^{\frac{2\pi i}{n}(k-1-n)}}{2} & m = 2k \end{cases} \\
&= \tau \circ \varphi_1 \circ \begin{cases} t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}k} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}k}+\mathrm{e}^{\frac{2\pi i}{n}(k+1)}}{2} & m = 2k+1 \\ t \mapsto (1-t)\mathrm{e}^{\frac{2\pi i}{n}k} + t\frac{\mathrm{e}^{\frac{2\pi i}{n}k}+\mathrm{e}^{\frac{2\pi i}{n}(k-1)}}{2} & m = 2k \end{cases} \\
&= \varphi \circ \pi_1(c_m).
\end{aligned}
$$

□

It is possible to transfer the notions of *interior* and *boundary* points from an n–gon to a twisted topological polygon.

Definition 5.1.13. *Let (P_f, π_f) be a twisted topological polygon with homeomorphism $\varphi_f : Q \to P_f$, where Q is a standard n—gon. We call $x \in P_f$ a **boundary point** if*

$$
\varphi_f^{-1}(x) = te^{\frac{2\pi i}{n}k} + (1-t)e^{\frac{2\pi i}{n}(k+1)}
$$

*for some integer $1 \leq k \leq n$ and $t \in [0,1]$. Otherwise, x is an **interior point**.*

Well–defined. By Definition 5.1.12, the map π_f maps a chamber to a path that consists only of boundary points. Furthermore, every boundary point lies on one of these paths. Thus, this definition is independent from the choice of strong polygon path. □

In order to construct the topological realisation, we first combine all of the twisted topological polygons into a large space, and then we construct an appropriate quotient topology.

Definition 5.1.14. *Let $P = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex. For any face $f \in F$, let (P_f, π_f) be a twisted topological polygon of f. The* ***twisted polygon flock of*** *P is the pair (U, π), where $U := \bigcup_{f \in F} P_f$ and $\pi : C \to U^{[0,1]}$ maps $c \in C$ to the path $\pi_{\lambda_2(c)}(c)$.*

At this point, the polygons of all faces are disjoint. We have to define an appropriate equivalence relation on them to "glue" them together. Since equivalent chambers should correspond to neighbouring polygons, we identify the path of these chambers.

But since there might be vertices with more than one maximal umbrella, we need to define an equivalence relation on them as well. Combining these ideas gives the full definition of the topological realisation.

Definition 5.1.15. *Let $P = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex. with twisted polygonal flock (U, π). The* ***topological realisation of*** *P is the quotient topology $U/\sim_U$, where $\sim_U$ is an equivalence relation on U defined by:*

- *If $c_1 \sim c_2$, we set $\pi(c_1)(t) \sim_U \pi(c_2)(t)$ for all $t \in [0, 1]$.*
- *If $\lambda_0(c_1) = \lambda_0(c_2)$, we set $\pi(c_1)(0) \sim_U \pi(c_2)(0)$.*

We mention without proof that the topological realisation can be described as a CW–complex.

Definition 5.1.15 is sensible, i. e. combinatorial properties are reflected in the topology.

Lemma 5.1.16. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex with twisted polygon flock (U, π) and topological realisation $U/\sim_U$.*

- *Let $p \in U$ be an interior point of one of the twisted topological polygons. Then, $|[p]_{\sim_U}| = 1$.*
- *Let $c \in C$ be a chamber with $\lambda_0(c) = v$. Then, $|[\pi(c)(0)]_{\sim_U}| = \deg(v)$.*
- *Let $c \in C$ be a chamber with $\lambda_1(c) = e$. Then, $|[\pi(c)(t)]_{\sim_U}| = |[c]_\sim|$ for all $0 < t \leq 1$.*

Proof. The first claim is obvious since $\sim_U$ is only non–trivial on the boundaries of the twisted topological polygons.

The third claim follows since the paths $\pi(c)$ for $c \in C$ are injective and $\sim_U$ identifies as many points as $|[c]_\sim|$.

For the second claim: For all $c \in C$ with $\lambda_0(c) = v$, the points $\pi(c)(0)$ are identified. Since $\pi(c)(0) = \pi(\sigma_1(c))(0)$, these come in pairs of two. Thus, we identify $\deg(v)$ many points. $\square$

At this point, we defined the topological realisation of a twisted polygonal complex. Actually, this defines a functor: We can associate a continuous map to each twisted polygonal morphism.

Lemma 5.1.17. *Let* $(\mu_V, \mu_E, \mu_F, \mu_C) : P^1 \to P^2$ *be a twisted polygonal morphism. Then, there is a continuous map between the topological realisations.*

Proof. We employ the notation $P^k = (V^k, E^k, F^k, C^k, \lambda^k, \sigma_0^k, \sigma_1^k, \sim^k)$ for $k \in \{1,2\}$.

For any face $f \in F^1$, we have $\mu_F(f) \in F^2$. Since a twisted polygonal morphism respects adjacencies, μ_C maps strong polygon paths of f to strong polygon paths of $\mu_F(f)$. Let $(c_1, c_2, \ldots, c_{2n})$ be a strong polygon path of f and (P_f, π_f) a twisted topological polygon based on this path. Then, $(\mu_C(c_1), \mu_C(c_2), \ldots, \mu_C(c_{2n})$ is a strong polygon path of $\mu_F(f)$, and $(P_f, \pi_{\mu_F(f)})$ with

$$\pi_{\mu_F(f)} : \{\mu_C(c_1), \mu_C(c_2), \ldots, \mu_C(c_{2n})\} \to P_f^{[0,1]} \qquad \mu_C(c_k) \mapsto \pi_f(c_k)$$

is a twisted topological polygon of $\mu_F(f)$. Thus, we have a homeomorphism from (P_f, π_f) to $(P_f, \pi_{\mu_F(f)})$. Combining them yields a continuous map $\mu : U^1 \to U^2$, where (U^k, π^k) is the twisted polygon flock of P^k for $k \in \{1,2\}$.

We have to show that μ induces a continuous map $U^1/\sim_{U^1} \to U^2/\sim_{U^2}$. Call the projection maps $\pi_{U^k} : U^k \to U^k/\sim_{U^k}$, then the induced map is $\pi_{U^2} \circ \mu \circ (\pi_{U^1})^{-1}$ (note that π_{U^1} is not invertible in general, so we have to show that this definition is independent from the choice of the preimage). First, we have to show that the induced map is well–defined.

Let $x, y \in U^1$ with $\pi_{U^1}(x) = \pi_{U^1}(y)$. There are two possibilities:

1. $x = \pi^1(c_1)(t)$ and $y = \pi^1(c_2)(t)$ with $0 < t \leq 1$. This is only possible for $c_1 \sim^1 c_2$. In this case, we have $\mu_C(c_1) \sim^2 \mu_C(c_2)$, which guarantees $\pi^2(\mu_C(c_1))(t) \sim_{U^2} \pi^2(\mu_C(c_2))(t)$.

2. $x = \pi^1(c_1)(0)$ and $y = \pi^1(c_2)(0)$ for $c_1, c_2 \in C^1$ with $\lambda_0(c_1) = \lambda_0(c_2)$. This is transferred by μ as well, so $\pi^2(\mu_C(c_1))(0) \sim_{U^2} \pi^2(\mu_C(c_2))(0)$.

Now, we have to show that it is continuous. Let $A \subseteq U^2/\sim_{U^2}$ be an open set. By Definition 5.1.8, the set of preimages $(\pi_{U^2})^{-1}(A)$ is open in U^2. Since μ is continuous, $\mu^{-1}((\pi_{U^2})^{-1}(A))$ is open in U^1. By Definition 5.1.8, this is equivalent to

$$(\pi_{U^1} \circ \mu^{-1} \circ (\pi_{U^2})^{-1})(A)$$

being open in $U^1/\sim_{U^1}$. Thus, the induced map is continuous. □

5.2 Connectivity and strong connectivity

In this section, we explore different notions of connectivity for combinatorial complexes. There are two different types of connectivity that we are concerned with.

- The first one, *connectivity*, is mostly topological. A combinatorial complex is connected if its topological realisation is path–connected. For polygonal complexes, it is equivalent to the connectivity of the vertex–edge–graph (Definition 3.2.1).

- The second one, *strong connectivity*, uses topological and combinatorial structure. In many cases, it is more natural to use. In the topological realisation, it corresponds to path–connectivity after removal of all vertices. For polygonal complexes, it is equivalent to the connectivity of the face–edge–graph (Definition 3.3.1).

  In the literature, it appeared e. g. in [49, Section 3] and [44, Section 2].

  We stress that *strong connectivity* is not related to the equally named concept for directed graphs (which can be found in [35, Section 1.1, D45]).

5.2.1 For twisted polygonal complexes

In this subsection, we define *connectivity* and *strong connectivity* for twisted polygonal complexes. We start with the formalisation of strong connectivity. It relies on the concept of strong paths (compare Definition 2.4.12 from Subsection 2.4.1).

Definition 5.2.1. *A twisted polygonal complex* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *is* ***strongly connected*** *if for any two faces* $f_1, f_2 \in F$ *there is a strong path* $(c_1, \dots, c_n) \in C^n$ *with* $\lambda_2(c_1) = f_1$ *and* $\lambda_2(c_2) = f_2$*.*

To define the weaker notion of *connectivity*, we need to define a different type of path.

Definition 5.2.2. *Let* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *be a twisted polygonal complex with chambers* $c_1 \neq c_2 \in C$*. For* $k \in \{0, 1, 2\}$*, the chambers* c_1 *and* c_2 *are* ***weakly*** k***–adjacent*** *if* $\lambda_k(c_1) = \lambda_k(c_2)$*.*

c_1 *and* c_2 *are* ***weakly adjacent*** *if they are weakly* k*–adjacent for at least one* $k \in \{0, 1, 2\}$*.*

If two chambers are strongly adjacent, they are also weakly adjacent.

Remark 5.2.3. *Let* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *be a twisted polygonal complex and* $c_1 \neq c_2 \in C$*. If* c_1 *and* c_2 *are* k*–adjacent for* $k \in Z := \{0, 1, 2\}$*, they are weakly* m*–adjacent for all* $m \in Z \backslash \{k\}$*.*

Proof. We check the different cases of adjacency (Definition 2.4.11):

- If c_1 and c_2 are 0–adjacent, we have $c_2 = \sigma_0(c_1)$. By Definition 2.4.1, σ_0 does not change the value of λ_1 and λ_2.
- If c_1 and c_2 are 1–adjacent, we have $c_2 = \sigma_1(c_1)$. By Definition 2.4.1, σ_0 does not change the value of λ_0 and λ_2.
- c_1 and c_2 are 2–adjacent, we have $c_1 \sim c_2$. By Definition 2.4.1, the values of λ_0 and λ_1 are identical for c_1 and c_2. □

In Subsection 2.5.2, strong paths are constructed from adjacent chambers. Here, we construct *weak paths* from weakly adjacent chambers.

Definition 5.2.4. *Let* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *be a twisted polygonal complex.*

A ***weak path*** *is a sequence* $(c_1, c_2, \dots, c_n) \in C^n$ *such that* c_i *and* c_{i+1} *are weakly adjacent for all* $1 \leq i < n$*.*

Definition 5.2.5. *A twisted polygonal complex* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *is* ***connected*** *if for any two faces* $f_1, f_2 \in F$ *there is a weak path* $(c_1, \dots, c_n) \in C^n$ *with* $\lambda_2(c_1) = f_1$ *and* $\lambda_2(c_2) = f_2$*.*

It is easy to see that strong connectivity is a stronger notion than connectivity.

Remark 5.2.6. *A strongly connected twisted polygonal complex is connected.*

Proof. By Remark 5.2.3, every strong path is also a weak path. □

The converse direction is not always true, but it becomes true if we restrict attention to twisted polygonal surfaces.

Lemma 5.2.7. *A connected twisted polygonal surface is strongly connected.*

Proof. Let f_1 and f_2 be two faces and consider a weak path $(c_1, \dots, c_n)$. If we can construct a strong path between the chambers c_k and c_{k+1} for all $1 \leq k < n$, we can combine them into a strong path from c_1 to c_2.

- If $\lambda_0(c_k) = \lambda_0(c_{k+1})$, both c_k and c_{k+1} lie in the same strong umbrella path. This is the only case where we need the assumption that we have a *surface*.
- If $\lambda_1(c_k) = \lambda_1(c_{k+1})$, Definition 2.4.1 implies $c_k \sim c_{k+1}$ or $c_k \sim \sigma_0(c_{k+1})$. In the first case, (c_k, c_{k+1}) is a strong path. In the second case, $(c_k, \sigma_0(c_{k+1}), c_{k+1})$ is a strong path.
- If $\lambda_2(c_k) = \lambda_2(c_{k+1})$, Definition 2.4.1 implies $c_{k+1} \in \langle \sigma_0, \sigma_1 \rangle . c_k$. In other words, $c_{k+1} = \tau_1 \tau_2 \cdots \tau_m . c_k$ with $\tau_i \in \{\sigma_0, \sigma_1\}$. Then,

$$(c_k, \tau_m . c_k, \tau_{m-1}\tau_m . c_k, \dots, \tau_2 \cdots \tau_m . c_k, c_{k+1})$$

  is a strong path. □

Now, we show the correspondence between (strong) connectivity of a twisted polygonal complex and the path–connectivity of its topological realisation.

Lemma 5.2.8. *A twisted polygonal complex is connected if and only if its topological realisation is path–connected.*

A twisted polygonal complex is strongly connected if and only if its topological realisation (without vertices) is path–connected.

Proof. Let $P = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be the twisted polygonal complex with twisted polygon flock $U = \biguplus_{f \in F} P_f$ and topological realisation $U/\sim_U$.

Assume P is connected. Let $x_1, x_2 \in U/\sim_U$. Since each twisted topological polygon is path–connected, we only have to consider the case that x_1 and x_2 lie in different polygons, say P_f and P_g. By Definition 5.2.5, there is a weak path $(c_1, \dots, c_n) \in C^n$ with $\lambda_2(c_1) = f$ and $\lambda_2(c_n) = g$.

Consider the pair c_k and c_{k+1} for $1 \leq k < n$.

- If $\lambda_0(c_k) = \lambda_0(c_{k+1})$, the twisted topological polygons $P_{\lambda_2(c_k)}$ and $P_{\lambda_2(c_{k+1})}$ share a point in $U/\sim_U$. Thus, there is a path between them.
- If $\lambda_1(c_k) = \lambda_1(c_{k+1})$, there is a chamber $c \in C$ with $\lambda_2(c) = \lambda_2(c_{k+1})$ and $c \sim c_k$ (by Definition 2.4.1). Thus, the twisted topological polygons of the two chambers share a point in $U/\sim_U$.

Thus, we can always find a path between any two points $x_1, x_2 \in X$.

Conversely, assume $U/\sim_U$ is path–connected. Consider two faces $f, g \in F$. Pick any point $x_1 \in P_f$ and $x_2 \in P_g$, then there is a path $p : [0,1] \to U/\sim_U$ with $p(0) = x_1$ and $p(0) = x_2$. This path has to traverse some twisted topological polygons. If it moves from P_{f_1} to P_{f_2}, there are two options to do so (with respect to $\sim_U$ from Definition 5.1.15):

- It can be an identification because of chambers $c_1 \sim c_2$ with $\lambda_2(c_k) = f_k$. In this case, we construct a path to c_1 (the chambers within a face can be connected by a strong path) and continue with c_2, which is strongly adjacent.
- It can be an identification because of chambers $c_1, c_2 \in C$, satisfying $\lambda_2(c_k) = f_k$ and $\lambda_0(c_1) = \lambda_0(c_2)$. In this case, construct a path to c_1 and continue with c_2, which is weakly adjacent.

This shows the equivalence between connectivity of P and path–connectivity of $U/\sim_U$. Furthermore, a close analysis of this proof reveals the stronger claim: P is strongly connected if and only if $U/\sim_U$ is path–connected after removal of all vertices. □

5.2.2 For polygonal complexes

In this subsection, we define *connectivity* and *strong connectivity* for polygonal complexes. We start with the formalisation of strong connectivity. It relies on the concept of strong paths (compare Definition 2.5.17 from Subsection 2.5.2.

Definition 5.2.9. *A polygonal complex (V, E, F, η, φ) is* ***strongly connected*** *if for any two faces $f, g \in F$ there is an edge–face–path $(e_0, f_1, e_1 \ldots, f_n, e_n)$ with $f_1 = f$ and $f_n = g$.*

The weaker notion of *connectivity* is defined with a different type of path.

Definition 5.2.10. *Let (V, E, F, η, φ) be a polygonal complex. A* ***vertex–edge–path*** *is a sequence $(v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$ such that*

- *$v_i \in V$ for all $0 \le i \le n$.*
- *$e_i \in E$ for all $1 \le i \le n$.*
- *v_{i-1} and v_i are incident to e_i for all $1 \le i \le n$.*

In the same form as the definition of strong connectivity, we define connectivity.

Definition 5.2.11. *A polygonal complex (V, E, F, η, φ) is* ***connected*** *if for any two vertices $v, w \in V$ there is a vertex–edge–path $(v_0, e_1, v_1, \dots, e_n, v_n)$ with $v_0 = v$ and $v_n = w$.*

Strong connectivity is a stronger notion than connectivity.

Lemma 5.2.12. *A strongly connected polygonal complex is connected.*

Proof. Let v, w be two vertices. By Definition 2.5.2, there are faces f, g such that $v \prec f$ and $w \prec g$. Since the polygonal complex is strongly connected, there is an edge–face–path $(e_0, f_1, e_1, \dots, f_n, e_n)$ with $f = f_1$ and $g = f_n$.

Without loss of generality, we choose the edges e_0 and e_n of the path in such a way that $v \prec e_0$ and $w \prec e_n$. Choose vertices $v_i \prec e_i$ for all $1 \leq i < n$. This gives the vertex sequence

$$(v, v_1, v_2, \dots, v_{n-1}, w)$$

Every two vertices that are adjacent in this sequence are incident to the same face. Since two vertices incident to the same face are always connected by a vertex–edge–path (the perimeter of the face), the claim follows. □

The converse direction is not always true, but becomes true if we restrict our attention to polygonal surfaces.

Lemma 5.2.13. *A connected polygonal surface is strongly connected.*

Proof. Let f, g be two faces. Choose vertices v, w with $v \prec f$ and $w \prec g$. Since the polygonal complex is connected, there is a vertex–edge–path $(v_0, e_1, v_1, \dots, e_n, v_n)$ with $v_0 = v$ and $v_n = w$.

Since v_0 is either an inner vertex or a boundary vertex, there is an umbrella–path from f to a face f_1 satisfying $e_1 \prec f_1$. Since $\prec$ is transitive, $v_1 \prec f_1$. Thus, we can repeat this argument to construct an edge–face–path from f to g. □

We note that both concepts of connectivity can be found in the associated graphs of a polygonal complex (compare Definition 3.2.1 and Definition 3.3.1).

Remark 5.2.14. *A polygonal complex is*

- *connected if and only if its vertex–edge–graph is connected.*
- *strongly connected if and only if its face–edge–graph is connected.*

5.2.3 For Dress–surfaces

In the previous subsection, we have seen that connectivity and strong connectivity do not differ for twisted polygonal surfaces (Remark 5.2.6 and Lemma 5.2.7) and polygonal surfaces (Lemma 5.2.12 and Lemma 5.2.13).

Since there is no separate concept of "combinatorial complex" for Dress–surfaces, we only define *strong connectivity*.

Definition 5.2.15. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface. It is called* ***(strongly) connected*** *if the group $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ acts transitively on C.*

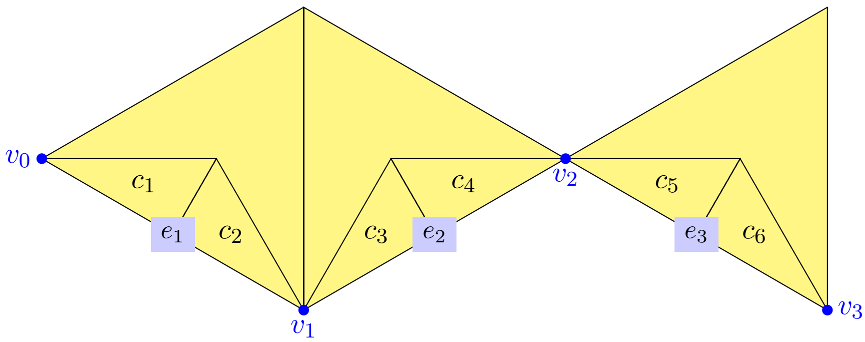
5.2.4 Compatibilities

The previous subsections define *connectivity* and *strong connectivity* for twisted polygonal complexes, polygonal complexes, and Dress–surfaces. In this subsection, we show that both concepts are combinatorial properties, i. e. they are preserved by the functors between the categories (compare Subsection 2.8.1).

We start with the functor $\mathsf{TwistPoly} : \mathbf{PolyComp} \to \mathbf{TwistPolyComp}$. It is useful to recall certain chambers which are always connected by a strong path.

Remark 5.2.16. *Let $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted polygonal complex and $c_1, c_2 \in C$. If $\lambda_1(c_1) = \lambda_1(c_2)$ or $\lambda_2(c_1) = \lambda_2(c_2)$ holds, there exists a strong path with first element c_1 and last element c_2.*

We construct a correspondence between the weak path $(c_1, c_2, \dots)$ in a twisted polygonal complex and the vertex–edge–path $(v_0, e_1, v_1, \dots)$ like in the following illustration:



Lemma 5.2.17. *Let P be a polygonal complex. Then, P is (strongly) connected if and only if $\mathsf{TwistPoly}(P)$ is (strongly) connected.*

Proof. We use the notation from Definition 2.5.13 for P and $\mathsf{TwistPoly}(P)$.

- Let P be connected. Let $f, g \in F$ and choose vertices $v, w \in V$ with $v \prec f$ and $w \prec g$. Since P is connected, there is a vertex–edge–path $(v_0, e_1, v_1, \dots, e_n, v_n)$ with $v_0 = v$ and $v_n = w$.

  Choose faces f_i with $e_k \prec f_k$ (for $1 \leq k \leq n$) satisfying $f_1 = f$ and $f_n = g$. Then, define

$$\begin{aligned} c_{2k-1} &:= (v_{k-1}, e_k, f_k) \in C \\ c_{2k} &:= (v_k, e_k, f_k) \in C. \end{aligned}$$

  Then, $(c_1, c_2, \dots, c_{2n-1}, c_{2n})$ is a weak path with $\lambda_2(c_1) = f_0 = f$ and $\lambda_2(c_{2n}) = f_n = g$.

- Let $\mathsf{TwistPoly}(P)$ be connected. Let $v, w \in V$ and choose faces $f, g \in F$ with $v \prec f$ and $w \prec g$. Since $\mathsf{TwistPoly}(P)$ is connected, there is a weak path $(c_1, \dots, c_n)$ with $\lambda_2(c_1) = f$ and $\lambda_2(c_n) = g$. If $\lambda_0(c_1) \neq v$, we extend the path by the chamber

(v, e, f), with the edge e chosen appropriately. We do the same for c_n. After these extensions, we have a weak path with $\lambda_0(c_1) = v$ and $\lambda_0(c_n) = w$.

If we can show that weak adjacency of c_k and c_{k+1} implies the existence of a vertex–edge–path from $\lambda_0(c_k)$ to $\lambda_0(c_{k+1})$, the claim is proven.

If $\lambda_0(c_k) = \lambda_0(c_{k+1})$, there is nothing to prove. If $\lambda_1(c_k) = \lambda_1(c_{k+1})$, the sequence $(\lambda_0(c_k), \lambda_1(c_k), \lambda_0(c_{k+1}))$ describes a vertex–edge–path. If $\lambda_2(c_k) = \lambda_2(c_{k+1})$, the vertices $\lambda_0(c_k)$ and $\lambda_0(c_{k+1})$ lie in the same polygon of P, so they are connected by a vertex–edge–path.

- Let P be strongly connected.

  For $f, g \in F$, there is an edge–face–path $(e_0, f_1, e_1, \ldots, f_n, e_n)$ with $f = f_1$ and $g = f_n$. Let c_k^- be any chamber with $\lambda_{12}(c_k^-) = (e_{k-1}, f_k)$ and c_k^+ any chamber with $\lambda_{12}(c_k^+) = (e_k, f_k)$.

  Since $\lambda_2(c_k^-) = \lambda_2(c_k^+)$, there is a strong path from c_k^- to c_k^+ (Remark 5.2.16). Since $\lambda_1(c_k^+) = \lambda_1(c_{k+1}^-)$, there is a strong path from c_k^+ to c_{k+1}^- (Remark 5.2.16). Combining them yields a strong path from c_1^- to c_n^+ that satisfies $\lambda_2(c_1^-) = f$ and $\lambda_2(c_n^+) = g$.

- Let $\mathsf{TwistPoly}(P)$ be strongly connected. For $f, g \in F$, there is a strong path $(c_1, \ldots, c_n)$ with $\lambda_2(c_1) = f$ and $\lambda_2(c_n) = g$.

  We partition the set $\{1, 2, \ldots, n\}$ into maximal disjoint intervals $I_1, I_2, \ldots, I_m$, with the stipulation that $\lambda_2(c_i) = \lambda_2(c_j)$ if $i, j \in I_k$. We define f_k as $\lambda_2(c_i)$ for any $i \in I_k$.

  Consider the chambers $c_{\max I_k}$ and $c_{\min I_{k+1}}$. Since they differ on λ_2, they have to be 2–adjacent. In particular, we can define e_k as their image under λ_1.

  We obtain an edge–face–path

  $$(\lambda_1(c_{\min I_1}), f_1, e_1, \ldots, f_k, \lambda_1(c_{\max I_k}))$$

  with $f_1 = f$ and $f_k = g$. □

Next, we consider the functor $\mathsf{TwistDress} : \mathbf{DressSurf} \to \mathbf{TwistPolyComp}$. The involutions σ_0 and σ_1 appear in both of them, while σ_2 only appears in the formalisation of Dress–surfaces. However, it is very tightly connected with the equivalence relation $\sim$ from the formalisation of twisted polygonal complexes. This can be seen in Definition 2.7.10.

Remark 5.2.18. *Let $S = (C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface. Then, $\sigma_2(c) \sim c$ for all chambers $c \in C$ in the twisted polygonal surface* $\mathsf{TwistDress}(S)$*.*

The central observation in the correspondence between Dress–surfaces and twisted polygonal surfaces is, that the action of an involution in $\{\sigma_0, \sigma_1, \sigma_2\}$ corresponds to shifting to an adjacent chamber.

Lemma 5.2.19. *Let S be a Dress–surface. Then, S is strongly connected if and only if* $\mathsf{TwistDress}(S)$ *is strongly connected.*

Proof. A Dress–surface $(C, \sigma_0, \sigma_1, \sigma_2)$ is strongly connected if and only if $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ acts transitively on C. Equivalently, for any two chambers $c_1, c_2 \in C$, we can write $c_2 = w.c_1$ for some $w \in \langle \sigma_0, \sigma_1, \sigma_2 \rangle$. Since all σ_k are involutions (implying $\sigma_k^{-1} = \sigma_k$), the element w can be written as $\tau_m \cdots \tau_2 \tau_1$ with $\tau_i \in \{\sigma_0, \sigma_1, \sigma_2\}$ for $1 \leq i \leq m$.

Taking Remark 5.2.18 into consideration, we can write this product as the strong path

$$(c_1, \tau_1.c_1, \tau_2\tau_1.c_1, \ldots, \tau_{m-1} \cdots \tau_1.c_1, c_2)$$

in $\mathsf{TwistDress}(S)$. Clearly, if there is a strong path between any two chambers, the twisted polygonal surface $\mathsf{TwistDress}(S)$ is strongly connected.

Conversely, let $\mathsf{TwistDress}(S)$ be strongly connected. Given two chambers $c, d \in C$, we obtain a strong path $(c_1, \ldots, c_n)$ with $\lambda_2(c) = \lambda_2(c_1)$ and $\lambda_2(d) = \lambda_2(c_n)$. Since two chambers in the same face are always connected by a strong path (Remark 5.2.3), we can extend this path to a strong path from c to d.

Without loss of generality, $c_k \neq c_{k+1}$ for $1 \leq k < n$. Then, we can represent this path as an element in the group $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$: Consider the adjacent chambers c_k and c_{k+1}.

- If they are 0–adjacent, we have $\sigma_0(c_k) = c_{k+1}$ by Definition 2.4.11.
- If they are 1–adjacent, we have $\sigma_1(c_k) = c_{k+1}$ by Definition 2.4.11.
- If they are 2–adjacent, we have $c_k \sim c_{k+1}$ by Definition 2.4.11. By combining $c_k \neq c_{k+1}$ with $c_k \sim c_{k+1}$, we obtain $\sigma_2(c_k) = c_{k+1}$.

This shows that $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ acts transitively on C. □

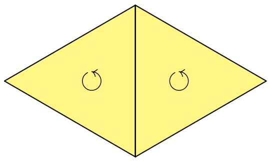
5.3 Orientability and dual orientability

In this section, we explore different concepts of orientation. Specifically, we analyse these two:

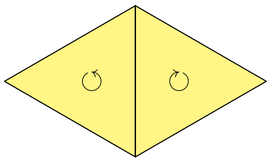
- *Orientability*, which is a discrete analogue of topological orientability.
- *Dual orientability*, which has no topological analogue, as it heavily relies on the combinatorial structure.

In both cases, we assign each face a “local orientation”. If we imagine the face as embedded polygon in $\mathbb{R}^3$, this corresponds to a choice of one of the sides of the polygon. We can illustrate this by a cyclic arrow:

A combinatorial surface is *orientable* if we can choose local orientations in such a way that the local orientations of two adjacent faces are compatible. In an embedding into $\mathbb{R}^3$, this corresponds to choosing "the same side" for two adjacent polygons. In the illustration with cyclic arrows, it corresponds to the following situation:



We define *dual orientation* by the opposite compatibility relation:



5.3.1 For polygonal surfaces

In this subsection, we define *orientation* and *dual orientation* for polygonal surfaces. To do so, we need to transform intuitive description from the start of Section 5.3 into rigorous mathematical statements.

For polygonal surface, we formalise the cycle arrow notation that we have used in the illustrations. We associate each face to a cyclic permutation of its vertices.

Definition 5.3.1. *Let (V, E, F, η, φ) be a polygonal complex. A* ***local orientation map*** *is a map $c : F \to \mathrm{Sym}(V)$ with the following property:*

- *Let $f \in F$, then there is an alternating sequence $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m)$ of vertices and edges incident to f (Definition 2.5.2). Then $c(f)$ is either the cycle $(v_1, v_2, \ldots, v_m)$ or the cycle $(v_m, \ldots, v_2, v_1)$.*

This concept allows us to define orientation and dual orientation.

Definition 5.3.2. *A polygonal surface (V, E, F, η, φ) is* ***orientable*** *if there is a local orientation map $c : F \to \mathrm{Sym}(V)$ such that*

- *For each inner edge with incident vertices v_1 and v_2 and incident faces f_1 and f_2, we have*
$$c(f_1)(v_1) = v_2 \Leftrightarrow c(f_2)(v_2) = v_1,$$

Intuitively, the cyclic permutations of the faces induce "opposite orientations" on the edges between them if they can be combined into a consistent global orientation. However, if all of them induce the same "edge orientation", this allows us to define a dual orientation.

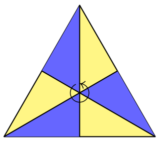
Definition 5.3.3. *A polygonal surface* (V, E, F, η, φ) *is* ***dual orientable*** *if there is a local orientation map* $c : F \to \mathrm{Sym}(V)$ *such that*

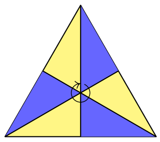
- *For each inner edge with incident vertices* v_1 *and* v_2 *and incident faces* f_1 *and* f_2*, we have*
$$c(f_1)(v_1) = v_2 \Leftrightarrow c(f_2)(v_1) = v_2,$$

5.3.2 For twisted polygonal surfaces

In this subsection, we define *orientation* and *dual orientation* for twisted polygonal surfaces. To do so, we need to transform the intuitive description from the start of Section 5.3 into rigorous mathematical statements.

We cannot use the approach for polygonal surfaces from Subsection 5.3.1 since vertices can be incident to a face "multiple times". We could define a cyclic permutation on the chambers that are incident to each face, but we choose a different formalisation that is based on the observation in this illustration:

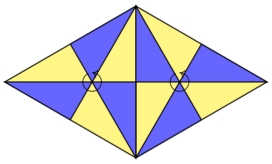




This means that we can encode the information "up or down" as a two–colouring of the chambers within a face. This is our primary formalisation for twisted polygonal surfaces.

Definition 5.3.4. *Let* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *be a twisted polygonal complex. A* ***local chamber colouring*** *is a map* $s : C \to \{\pm 1\}$ *such that two chambers have different values if they are 0– or 1–adjacent.*

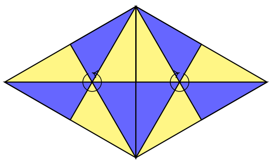
To define orientation, we have to formalise the compatibility between two adjacent faces. Visually, this gives the following illustration:



Thus, the value of s has to be different for chambers that are 2–adjacent.

Definition 5.3.5. *A twisted polygonal surface* $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ *is* ***orientable*** *if there is a local chamber colouring* $s : C \to \{\pm 1\}$ *such that 2–adjacent chambers have distinct values.*

For dual orientability, we reach the opposite conclusion:



Definition 5.3.6. *A twisted polygonal surface $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ is **orientable** if there is a local chamber colouring $s : C \to \{\pm 1\}$ such that 2–adjacent chambers have equal values.*

5.3.3 For Dress–surfaces

In this subsection, we define *orientation* and *dual orientation* for Dress–surfaces. To do so, we need to transform intuitive description from the start of Section 5.3 into rigorous mathematical statements.

We could choose the same formalisation as in Subsection 5.3.2 for twisted polygonal surfaces. However, we would waste the potential of formulating orientability group–theoretically. We start with the same visualisation of local orientation as Subsection 5.3.2.

Consider a Dress–surfaces $(C, \sigma_0, \sigma_1, \sigma_2)$ and two chambers c_1 and c_2 in the same strongly connected component. By Definition 5.2.15, there is an element $g \in \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ such that $c_2 = g.c_1$.

Instead of assigning $\{\pm 1\}$ to c_1 and c_2, we could assign the *change* of this assignment to g. Formally, this corresponds to the construction of a group homomorphism $\mu : \langle \sigma_0, \sigma_1, \sigma_2 \rangle \to \{\pm 1\}$ (we interpret the "shift" multiplicatively).

For both orientability and dual orientability, we would need $\mu(\sigma_0) = \mu(\sigma_1) = -1$. To encode orientability, σ_2 has to be mapped to -1 as well. To encode dual orientability, σ_2 has to be mapped to $+1$.

Although this idea sounds nice (we just need to check whether such a group homomorphism exists), it is not sufficient to formalise (dual) orientability. In particular, consider the stabiliser $\mathrm{Stab}_{\langle \sigma_0, \sigma_1, \sigma_2 \rangle}(c)$ of an arbitrary chamber c. Clearly, any element from this stabiliser should be mapped to $+1$. In other words,

$$\mathrm{Stab}_{\langle \sigma_0, \sigma_1, \sigma_2 \rangle}(c) \leq \ker(\mu)$$

for all chambers $c \in C$.

Definition 5.3.7. *The Dress–surface $(C, \sigma_0, \sigma_1, \sigma_2)$ is **orientable** if the map*

$$\mu : \langle \sigma_0, \sigma_1, \sigma_2 \rangle \to \langle -1 \rangle \qquad \sigma_0 \mapsto -1 \qquad \sigma_1 \mapsto -1 \qquad \sigma_2 \mapsto -1$$

extends to a well–defined group homomorphism that satisfies $\mathrm{Stab}_{\langle \sigma_0, \sigma_1, \sigma_2 \rangle}(c) \leq \ker(\mu)$ for all chambers $c \in C$.

Definition 5.3.8. *The Dress–surface $(C, \sigma_0, \sigma_1, \sigma_2)$ is **dual orientable** if the map*

$$\mu : \langle \sigma_0, \sigma_1, \sigma_2 \rangle \to \langle -1 \rangle \qquad \sigma_0 \mapsto -1 \qquad \sigma_1 \mapsto -1 \qquad \sigma_2 \mapsto +1$$

extends to a well–defined group homomorphism that satisfies $\mathrm{Stab}_{\langle \sigma_0, \sigma_1, \sigma_2 \rangle}(c) \leq \ker(\mu)$ for all chambers $c \in C$.

5.3.4 Compatibilities

In the previous subsections, we defined the concepts of *orientability* and *dual orientability* for polygonal surfaces, twisted polygonal surfaces, and Dress–surfaces. In this subsection, we show that both of these are combinatorial properties (in the sense of Subsection 2.8.1).

We start with the functor TwistPoly. The proof idea was informally presented in the motivation of Subsection 5.3.2.

Lemma 5.3.9. *The polygonal surface S is (dual) orientable if and only if* $\mathsf{TwistPoly}(S)$ *is (dual) orientable.*

Proof. Assume $S = (V, E, F, \eta, \varphi)$ is (dual) orientable with local orientation map $c : F \to \mathrm{Sym}(V)$. We define the local chamber colouring $s : C \to \{\pm 1\}$ as follows: Let $f \in F$ be a face. By Definition 5.3.1, there is an alternating sequence $(v_1, e_1, v_2, e_2, \dots, v_m, e_m)$ of vertices and edges incident to f such that $c(f) = (v_1, \dots, v_m)$. Define

$$s(v_k, e_k, f) := 1 \qquad\qquad s(v_k, e_{k-1}, f) := -1.$$

Clearly, two chambers that are 0–adjacent or 1–adjacent take different values under s, so this is a well–defined local chamber colouring. Consider two 2–adjacent chambers $c_1 = (v, e, f)$ and $c_2 = (v, e, g)$. Let the alternating vertex–edge–sequence of f be $(v, e, w, \dots)$ for some vertex w.

Now, there are two options for the alternating vertex–edge–sequence of g. It can have the form $(v, e, w, \dots)$ or $(\dots, w, e, v)$. In the first case, we have $c(g)(v) = w$, in the second one we obtain $c(g)(w) = v$. Consequently:

- If S is orientable, Definition 5.3.2 tells us that we are in the second case. Then, the 2–adjacent flags (v, e, f) and (v, e, g) have different values under s.
- If S is dual orientable, Definition 5.3.3 tells us that the first case applies. Then, the 2–adjacent flags (v, e, f) and (v, e, g) have the same value under s.

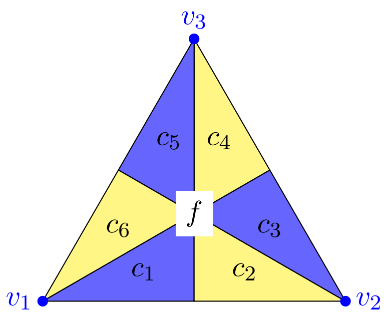
Conversely, assume $\mathsf{TwistPoly}(S)$ is (dual) orientable with local chamber map $s : C \to \{\pm 1\}$. We define the local orientation map $c : F \to \mathrm{Sym}(V)$ as follows: For any face $f \in F$, consider the chambers

$$C_f := \{c \in C \mid \lambda_2(c) = f \text{ and } s(c) = 1\}.$$

Since 1–adjacent chambers have different s–values, the map

$$\kappa : \{v \in V \mid v \prec f\} \to C_f \qquad\qquad v \mapsto c \text{ with } C_f \cap \lambda_0^{-1}(v) = \{c\}$$

is well–defined.



In the example on the left, we have

$$s(c_k) = \begin{cases} +1 & k \text{ odd} \\ -1 & k \text{ even.} \end{cases}$$

Then, $C_f = \{c_1, c_3, c_5\}$ and

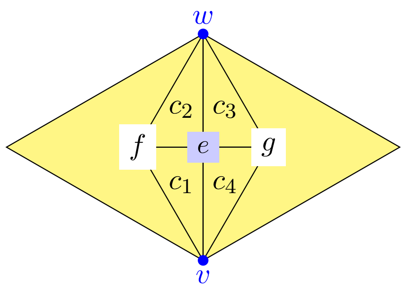
$$\kappa(v_1) = c_1, \quad \kappa(v_2) = c_3, \quad \kappa(v_3) = c_5.$$

We define $c(f) \in \mathrm{Sym}(V)$ as follows:

- If $v \not\prec f$, it stays fixed.
- If $v \prec f$, we map v to $\lambda_0(c)$, where c is the unique chamber that is 0–adjacent to $\kappa(v)$.

This makes $c(f)$ into a cycle. Consider an inner edge $e \in E$ with incident vertices $v, w \in V$ and incident faces $f, g \in F$. Denote the four chambers in $\lambda_1^{-1}(e)$ as follows:

$$\lambda(c_1) = (v, e, f), \qquad \lambda(c_2) = (w, e, f), \qquad \lambda(c_3) = (w, e, g), \qquad \lambda(c_4) = (v, e, g).$$



Without loss of generality, assume $c(f)(v) = w$, so $s(c_1) = 1$ and $s(c_2) = -1$.

- If $\mathsf{TwistPoly}(S)$ is orientable, we have $s(c_3) = 1$ and $s(c_4) = -1$, implying $c(g)(w) = v$. Thus, S is orientable.
- If $\mathsf{TwistPoly}(S)$ is dual orientable, we have $s(c_3) = -1$ and $s(c_4) = 1$, implying $c(g)(v) = w$. Thus, S is dual orientable. $\square$

Next, we consider the functor $\mathsf{TwistDress}$. The correspondence proof follows along the lines in which we motivated the definitions for Dress–surfaces in Subsection 5.3.3.

Lemma 5.3.10. *The Dress–surface S is (dual) orientable if and only if* $\mathsf{TwistDress}(S)$ *is (dual) orientable.*

Proof. We employ the notation from Definition 2.7.10 for S and $\mathsf{TwistDress}(S)$.

Assume S is (dual) orientable with group homomorphism μ. We have to define an appropriate local chamber colouring $s : C \to \{\pm 1\}$ of $\mathsf{TwistDress}(S)$. For each strongly connected component of $\mathsf{TwistDress}(S)$, we choose a chamber c and define $s(c) := 1$. Each other chamber c^* in the same connected component can be written as $c^* = g.c$ with $g \in \langle \sigma_0, \sigma_1, \sigma_2 \rangle$, and we define $s(c^*) := \mu(g)$. This map is well–defined, since any $g \in \mathrm{Stab}_{\langle \sigma_0, \sigma_1, \sigma_2 \rangle}(c^*)$ satisfies $\mu(g) = 1$. It is easy to see that s satisfies Definition 5.3.5 if S is orientable, and Definition 5.3.6 if S is dual orientable.

Conversely, assume $\mathsf{TwistDress}(S)$ is (dual) orientable with local chamber colouring $s : C \to \{\pm 1\}$. We need to show that

$$\mu : \langle \sigma_0, \sigma_1, \sigma_2 \rangle \to \langle -1 \rangle \qquad \sigma_0 \mapsto -1 \qquad \sigma_1 \mapsto -1 \qquad \sigma_2 \mapsto a$$

extends to a well–defined group homomorphism with $\mathrm{Stab}_{\langle \sigma_0, \sigma_1, \sigma_2 \rangle}(c) \leq \ker(\mu)$ for all chambers $c \in C$ (where $a = -1$ in the orientable case, and $a = +1$ in the dual orientable case).

Let $\tau_m \tau_{m-1} \cdots \tau_1 = id_C$, with $\tau_k \in \{\sigma_0, \sigma_1, \sigma_2\}$ for all $1 \leq k \leq m$, then this corresponds to the strong path

$$(c, \tau_1.c, \tau_2\tau_1.c, \ldots, \tau_{m-1} \cdots \tau_1.c, c)$$

in $\mathsf{TwistDress}(S)$ for any chamber $c \in C$. If we manage to show that any strong path that starts and ends at the same chamber corresponds to an element $g \in \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ with $\mu(g) = 1$, the map μ extends to a group homomorphism and the stabiliser condition holds as well.

In any case, this strong path has to have an even number of sign changes (since $\mathsf{TwistDress}(S)$ is (dual) orientable). Thus, $\mu(g) = 1$ and the (dual) orientability of S follows. $\square$

6 Extensions of vertex–faithful simplicial surfaces

In this chapter, we focus exclusively on vertex–faithful simplicial surfaces, which can also be described as simplicial complexes (for details, we refer to Subsection 2.7.1).

We are mainly interested in surfaces with boundary. In Section 6.1, we discuss under which circumstances it is possible to remove a boundary vertex such that the resulting object remains a simplicial surface.

In Section 6.2, we discuss three possible ways to extend a surface along its boundary. We perform this both for simplicial surfaces and for extended simplicial surfaces.

6.1 Removing a boundary vertex

In this section, we discuss in which situations a boundary vertex can be removed "safely".

To remove a boundary vertex from a simplicial surface, we have to know how it affects the incidence structure (before we can check whether it fulfils the conditions of Definition 2.5.2 and Definition 2.5.27).

Definition 6.1.1. *Let $S = (V, E, F, \eta, \varphi)$ be a simplicial surface with vertex $v \in V$. Then, S^{-v} is the quintuple $(V^{-v}, E^{-v}, F^{-v}, \eta^{-v}, \varphi^{-v})$ with*

- $V^{-v} := V \backslash \{v\}$.
- $E^{-v} := E \backslash \{e \in E \mid v \in \eta(e)\}$.
- $F^{-v} := F \backslash \{f \in F \mid v \in (\eta \between \varphi)(f)\}$.
- $\eta^{-v} := \eta_{|E^{-v}}$.
- $\varphi^{-v} := \varphi_{|F^{-v}}$.

This modified incidence structure can be defined very generally, but S^{-v} does not have to be a simplicial surface, as Example 6.1.2 and Example 6.1.3 show.

Example 6.1.2. *Consider the simplicial surface $S = (V, E, F, \eta, \varphi)$ with*

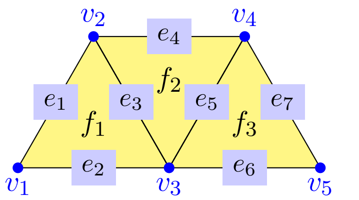
$$V = \{v_1, \dots, v_5\}, \qquad E = \{e_1, \dots, e_7\}, \qquad F = \{f_1, f_2, f_3\},$$

and

$$\eta : E \to \mathrm{Pot}_2(V) \qquad e_i \mapsto \begin{cases} \{v_i, v_{i+1}\} & 2 \nmid i \\ \{v_{i-1}, v_{i+1}\} & 2 \mid i \end{cases}$$

$$\varphi : F \to \mathrm{Pot}_3(E) \qquad f_i \mapsto \{e_{2i-1}, e_{2i}, e_{2i+1}\},$$

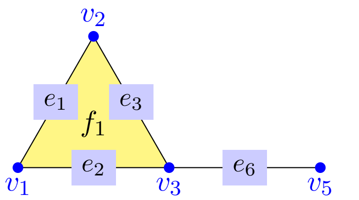
that can be illustrated like this:



Then, S^{-v_4} has the components

$$V^{-v_4} = \{v_1, v_2, v_3, v_5\}, \qquad E^{-v_4} = \{e_1, e_2, e_3, e_6\}, \qquad F^{-v_4} = \{f_1\},$$

with the illustration:



The illustration suggests that the edge e_6 is not incident to any face. Since $\varphi(f_1) = \{e_1, e_2, e_3\}$ and f_1 is the only element of F^{-v_4}, this is indeed the case. Therefore, S^{-v_4} is not a polygonal complex.

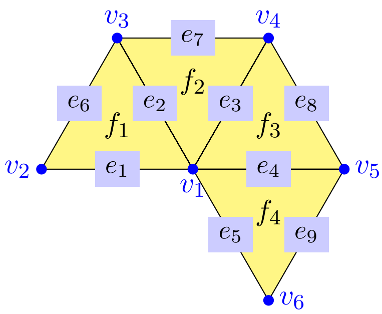
Example 6.1.3. *Consider the simplicial surface $S = (V, E, F, \eta, \varphi)$ with*

$$V = \{v_1, \ldots, v_6\}, \qquad E = \{e_1, \ldots, e_9\}, \qquad F = \{f_1, \ldots, f_4\},$$

and

$$\begin{aligned} \eta : E &\to \mathrm{Pot}_2(V) & e_i &\mapsto \begin{cases} \{v_1, v_{i+1}\} & i \leq 5 \\ \{v_{i-4}, v_{i-3}\} & i > 5 \end{cases} \\ \varphi : F &\to \mathrm{Pot}_3(E) & f_i &\mapsto \{e_i, e_{i+1}, e_{i+5}\}, \end{aligned}$$

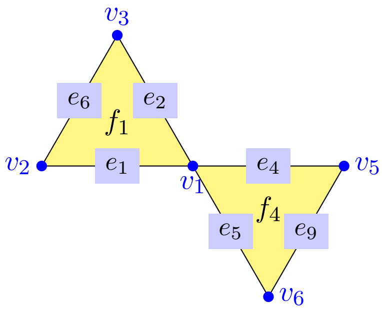
that can be illustrated like this:



Then, S^{-v_4} has the components

$$V^{-v_4} = \{v_1, v_2, v_3, v_5, v_6\}, \qquad E^{-v_4} = \{e_1, e_2, e_4, e_5, e_6, e_9\}, \qquad F^{-v_4} = \{f_1, f_4\},$$

with the illustration:



The illustration suggests that the vertex v_1 is ramified. Indeed, (e_1, f_1, e_2) and (e_4, f_4, e_5) are umbrella–paths around v_1. Since $\varphi(f_1) = \{e_1, e_2, e_6\}$ and $\varphi(f_4) = \{e_4, e_5, e_9\}$, both of these are maximal. Therefore, S^{-v_4} cannot be a polygonal surface.

To avoid these corner cases, we need to consider the vertices and edges "around" the boundary vertex. Formally, we talk about the *link* of a vertex. For simplicial complexes, this is a well–known concept, compare [47, Definition 2.13].

Definition 6.1.4. *Let $S = (V, E, F, \eta, \varphi)$ be a triangular complex and $v \in V$ a vertex. The* ***link*** $\mathrm{Lk}_S(v)$ *of v in S is the graph $(V_v, E_v, \eta_{|E_v})$, with*

$$V_v := \{w \in V \mid \exists e \in E : \eta(e) = \{v, w\}\}$$
$$E_v := \{e \in E \mid \exists f \in f : e \in \varphi(f) \wedge v \in (\eta \between \varphi)(f) \wedge v \notin \eta(e)\}.$$

Well–defined. We have to show that $(V_v, E_v, \eta_{|E_v})$ actually defines a graph according to Definition 3.1.1.

Let $e \in E_v$ and $\eta(e) = \{w_1, w_2\}$. By definition of E_v, there is an $f \in F$ with $e \in \varphi(f)$ and $v \in (\eta \between \varphi)(f)$. Combining these (and $|f| = 3$), we obtain $(\eta \between \varphi)(f) = \{v, w_1, w_2\}$. By Corollary 2.5.5, there are edges $e_1, e_2 \in E$ with $\eta(e_k) = \{v, w_k\}$ for $k \in \{1, 2\}$. ☐

We want to generalise the "problematic" vertices from Example 6.1.2 and Example 6.1.3. In both cases, we remove a boundary vertex that is connected to a different boundary vertex by an inner edge. We call these vertices *critical*.

Definition 6.1.5. *Let $S = (V, E, F, \eta, \varphi)$ be a simplicial surface. A vertex $v \in V$ is called* ***critical****, if there is a vertex $w \in V_v$ in the link with:*

- *w is a boundary vertex of S.*
- *There is an inner edge $e \in E$ with $\eta(e) = \{v, w\}$.*

Otherwise, v is called ***non–critical****.*

Example 6.1.6. *In the simplicial surface of Example 6.1.2, the critical vertices are* $\{v_2, v_3, v_4\}$.

In the simplicial surface of Example 6.1.3, the critical vertices are $\{v_1, v_3, v_4, v_5\}$.

At this point, it is not yet clear whether our concept of *critical vertices* captures the relevant situation. This is proven in the next lemma.

Lemma 6.1.7. *Let S be a vertex–faithful connected simplicial surface with more than one face. Let v be a non–critical vertex of S and $\mathrm{Lk}_S(v) = (V_v, E_v, \eta_v)$. Then, S^{-v} is a simplicial surface with degrees*

$$\deg_{S^{-v}}(w) = \begin{cases} \deg_S(w) & w \notin V_v \\ \deg_S(w) - 1 & w \in V_v \textit{ and } w \textit{ is a boundary vertex of } S \\ \deg_S(w) - 2 & w \in V_v \textit{ and } w \textit{ is an inner vertex of } S. \end{cases}$$

Furthermore, the inclusion map $S^{-v} \to S$ is a polygonal twilight morphism.

Proof. Denote $S = (V, E, F, \eta, \varphi)$ and $S^{-v} = (V^{-v}, E^{-v}, F^{-v}, \eta^{-v}, \varphi^{-v})$. We start by proving that S^{-v} is a triangular complex, according to Definition 2.5.2.

1. Since η^{-v} and φ^{-v} are restrictions of η and φ, such that $\varphi^{-v}(f) \subseteq E^{-v}$ for $f \in F^{-v}$, all faces remain polygons. In particular, they remain triangular.

2. Let $w \in V^{-v} \subseteq V$. Since S is vertex–faithful, there can be at most one edge $e \in E$ with $\eta(e) = \{v, w\}$. Since there are at least two edges incident to each vertex, there is an $e \in E$ with $w \in \eta(e)$ and $v \notin \eta(e)$, so $e \in E^{-v}$.

3. Let $e \in E^{-v} \subseteq E$.
   - If e is a boundary edge of S, there is exactly one face $f \in F$ with $e \in \varphi(f)$. We have to show that $v \notin (\eta \between \varphi)(f)$.

     Assume to the contrary that $v \in (\eta \between \varphi)(f)$. Let $w \in \eta(e)$, so $w \in V_v$. Since e is a boundary edge, w is a boundary vertex. By assumption, v is non–critical, so the edge connecting v and w has to be a boundary edge. Since this is true for all $w \in \eta(e)$, all edges in $\varphi(f)$ are boundary edges. Since S is connected, this implies $|F| = 1$, in contradiction to our assumption on S.
   - If e is an inner edge of S, let $\eta(e) = \{w_1, w_2\}$. Since S is vertex–faithful, there can be at most one face $f \in F$ with $(\eta \between \varphi)(f) = \{v, w_1, w_2\}$. Since e is an inner edge, there are two faces $f \in F$ with $e \in \varphi(f)$. Thus, one of them lies in F^{-v}.

Next, we show that S^{-v} is a simplicial surface according to Definition 2.5.27.

- Since for every $e \in E^{-v} \subseteq E$

  $$|\{f \in F^{-v} \mid e \in \varphi(f)\}| < |\{f \in F \mid e \in \varphi(f)\}|,$$

  there are no ramified edges.

- Let $w \in V^{-v} \subseteq V$. If $w \notin V_v$, its maximal umbrellas are unchanged, so it remains not ramified.

  Consider $w \in V_v$. If w is an inner vertex of S, then v is incident to exactly two faces in the umbrella. Removing them still leaves exactly one maximal umbrella. If w is a boundary vertex of S, there is a boundary edge connecting v and w (since v is non–critical). Therefore, v is incident to exactly one face of the maximal umbrella, whose removal does not split the umbrella.

The second analysis also proves the statement concerning the degrees of S^{-v}.

Finally, we have to show that the inclusion $S^{-v} \to S$ is a polygonal twilight morphism. Clearly, it is a polygonal morphism. To show that it is also a polygonal shadow morphism, we prove the contraposition of Definition 2.7.8.

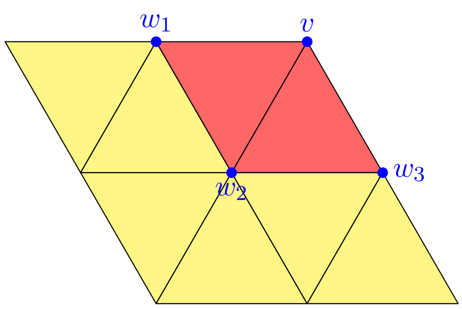
Let $x \in \mathrm{Pot}(V^{-v})$.

- If $x = \{v'\}$ for some $v' \in V$, then $v' \in V^{-v}$.
- If there is an edge $e \in E$ with $\eta(e) = x$, we have $e \in E^{-v}$ since $v \notin x$.
- If there is a face $f \in F$ with $(\eta \between \varphi)(f) = x$, we have $f \in F^{-v}$ since $v \notin x$.

Thus, the inclusion is a polygonal shadow morphism, and therefore also a polygonal twilight morphism. $\square$

Lemma 6.1.7 shows when we can remove a vertex from a simplicial surface. Since we are also interested in extended simplicial surface (to apply them in Chapter 8), we would like to extend this result to extended simplicial surfaces.

The main construction work has been done in Lemma 6.1.7 already. We only need to define the external degrees appropriately. Consider the following illustration of a simplicial surface, where the faces marked red should be removed.



To determine the external degrees of the surface S^{-v}, we count how many faces are removed at each vertex. For w_1 and w_2, one face is removed. For w_2, two faces are removed. We codify these observations in a lemma.

Lemma 6.1.8. *Let $(S, \widehat{\deg})$ be a vertex–faithful connected extended simplicial surface with more than one face. Let v be a non–critical vertex of S with $\mathrm{Lk}_S(v) = (V_v, E_v, \eta_v)$.*

Then, $(S^{-v}, \widehat{\deg}^{-v})$ *is an extended simplicial surface, with*

$$\widehat{\deg}^{-v}(w) = \begin{cases} \widehat{\deg}(w) + 1 & w \in V_v \text{ and } w \text{ is a boundary vertex of } S \\ 2 & w \in V_v \text{ and } w \text{ is an inner vertex of } S \\ \widehat{\deg}(w) & w \notin V_v. \end{cases}$$

Furthermore, $(S^{-v}, \widehat{\deg}^{-v}) \to (S, \widehat{\deg})$ *is an extended twilight morphism.*

Proof. From Lemma 6.1.7, we can deduce that S^{-v} is a simplicial surface and $S^{-v} \to S$ is a twilight morphism.

We have to show that $(S^{-v}, \widehat{\deg}^{-v})$ defines an extended simplicial surface according to Definition 4.2.1. For all vertices $w \notin V_v$, nothing changes. All vertices from V_v are boundary vertices in S^{-v}, and $\widehat{\deg}^{-v}$ is positive for them.

Finally, we have to show that $\iota : (S^{-v}, \widehat{\deg}^{-v}) \to (S, \widehat{\deg})$ is an extended twilight morphism according to Definition 4.2.10. Let w be a vertex of S^{-v}.

- If $w \notin V_v$, we have $\deg_{S^{-v}}(w) = \deg(w)$ and $\widehat{\deg}^{-v}(w) = \widehat{\deg}(w)$.
- If $w \in V_v$ and w is a boundary vertex of S,

$$\widehat{\deg}^{-v}(w) + \deg_{S^{-v}}(w) = \widehat{\deg}(w) + 1 + \deg(w) - 1 = \widehat{\deg}(w) + \deg(w).$$

- If $w \in V_v$ and w is an inner vertex of S,

$$\widehat{\deg}^{-v}(w) + \deg_{S^{-v}}(w) = 2 + \deg(w) - 2 = \widehat{\deg}(w) + \deg(w),$$

  since $\widehat{\deg}(w) = 0$.

Thus, ι is an extended twilight morphism. $\square$

6.2 Boundary extensions

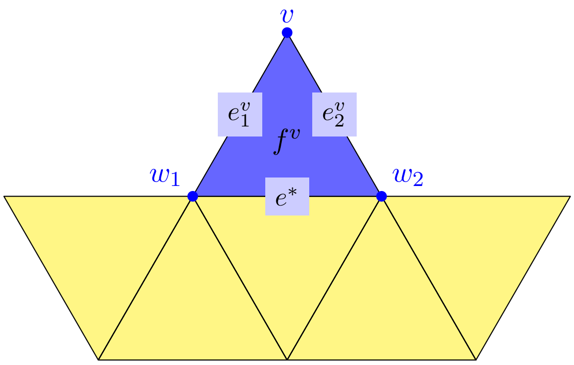
In Section 6.1, we removed a boundary vertex from a simplicial surface to create a new surface. In this section, we are doing the converse operation: *extending* a simplicial surface along its boundary.

We could describe a multitude of constructions here, but we restrict our attention to those three that are crucial for the constructions in Chapter 8. The proofs for these constructions can be transferred easily to other extensions not covered here.

Each subsection is dedicated to a specific extension. These are performed for simplicial surfaces and for extended simplicial surfaces. In the latter case, we also show in which cases the modified surface remains growth–controlled, assuming the starting surface is already growth–controlled (see Definition 4.2.4 for the definition of *growth–controlled*).

6.2.1 Extension by one face

In this subsection, we consider extensions of simplicial surfaces by one face. We want to model the scenario illustrated here (where the blue triangle is added):



We also show that this is the unique extension by one face, such that the edge e^* becomes an inner edge.

Lemma 6.2.1. *Let $S = (V, E, F, \eta, \varphi)$ be a simplicial surface and $e^* \in E$ a boundary edge of S. There exists a unique simplicial surface T with boundary vertex v such that*

1. $S = T^{-v}$.
2. $\deg_T(v) = 1$.
3. *e^* is an inner edge in T.*

With $\eta(e^*) = \{w_1, w_2\}$, it is given by $T = (V^+, E^+, F^+, \eta^+, \varphi^+)$, with

$$V^+ = V \cup \{v\}, \qquad E^+ = E \cup \{e_1^v, e_2^v\}, \qquad F^+ = F \cup \{f^v\},$$

and

$$\eta^+ : E^+ \to \mathrm{Pot}_2(V^+) \qquad e \mapsto \begin{cases} \eta(e) & e \in E \\ \{w_1, v\} & e = e_1^v \\ \{w_2, v\} & e = e_2^v. \end{cases}$$

$$\varphi^+ : F^+ \to \mathrm{Pot}_3(E^+) \qquad f \mapsto \begin{cases} \varphi(f) & f \in F \\ \{e^*, e_1^v, e_2^v\} & f = f^v. \end{cases}$$

We also have

$$\deg_T(w) = \begin{cases} \deg_S(w) & w \neq v \text{ and } w \notin \eta(e^*) \\ \deg_S(w) + 1 & w \in \eta(e^*) \\ 1 & w = v. \end{cases}$$

Proof. We start by showing the uniqueness. Let $T = (V^+, E^+, F^+, \eta^+, \varphi^+)$ be any such extension. From $T^{-v} = S$ we deduce $V^+ \backslash \{v\} = V$, so $V^+ = V \cup \{v\}$, with $v \notin V$.

The condition $\deg_T(v) = 1$ implies

$$|\{f \in F^+ \mid v \in (\eta^+ \between \varphi^+)(f)\}| = 1 \text{ and}$$
$$|\{e \in E^+ \mid v \in \eta^+(e)\}| = 2.$$

We denote the unique $f \in F^+$ from the first set by f^v. The two edges from the second set are called e_1^v and e_2^v.

Since e^* is an inner edge in T, but a boundary edge in T^{-v}, we know $e^* \in \varphi^+(f^v)$. Since $|\varphi^+(f^v)| = 3$, we conclude $\varphi^+(f^v) = \{e^*, e_1^v, e_2^v\}$. With $\eta(e^*) = \{w_1, w_2\}$ we have $(\eta^+ \between \varphi^+)(f^v) = \{v, w_1, w_2\}$. Without loss of generality, $\eta^+(e_k^v) = \{v, w_k\}$ for $k \in \{1, 2\}$.

Now we show that this construction actually defines a simplicial surfaces. Clearly, f^v is a polygon. Since all new vertices and edges are incident to an edge or a face, respectively, T is a triangular complex.

Since e^* is an inner edge, e_k^v are boundary edges, and all other edges are unchanged, there are no ramified edges in T.

Consider the vertices:

- v is a boundary vertex since (e_1^v, f_v, e_2^v) is a maximal umbrella around it.
- All vertices in $V \backslash \{w_1, w_2\}$ are unchanged.
- The vertex w_k (for $k \in \{1, 2\}$) has the maximal umbrella $(e_1, f_1, \ldots, e_n, f_n, e^*)$ in S. In T, this umbrella can be extended to $(e_1, f_1, \ldots, e_n, f_n, e^*, f^v, e_k^v)$, so w_k remains a boundary vertex.

Thus, there are no ramified vertices, which shows that T is a simplicial surface. □

The extension constructed in Lemma 6.2.1 deserves a name.

Definition 6.2.2. *We refer to the unique simplicial surface from Lemma 6.2.1 by $S_{e^*}^{+v}$.*

Next, we generalise the extension from Lemma 6.2.1 to extended simplicial surface, analogous to Lemma 6.1.8. However, there is some ambiguity: Since the vertex v is "new", we have no information about its external degree. To determine it uniquely, we need further restrictions. In our case, the construction in Chapter 8 requires the extended polygonal twilight morphisms to be *hexagonal* (compare Definition 4.2.11). This determines the external degree of v uniquely.

Lemma 6.2.3. *Let $(S, \widehat{\deg})$ be an extended simplicial surface and e a boundary edge of S, such that $\widehat{\deg}(w) > 1$ for all $w \in \eta(e)$. Then, $(S_e^{+v}, \widehat{\deg}_d^{+v})$ (with $d \in \mathbb{Z}_{\geq 1}$) is an extended simplicial surface with*

$$\widehat{\deg}_d^{+v}(w) = \begin{cases} \widehat{\deg}(w) & w \neq v \text{ and } w \text{ not incident to } e \\ \widehat{\deg}(w) - 1 & w \text{ incident to } e \\ d & w = v. \end{cases}$$

Furthermore, $(S, \widehat{\deg}) \to (S_e^{+v}, \widehat{\deg}_d^{+v})$ *is an extended twilight morphism. It is hexagonal if and only if* $d = 5$*.*

Proof. By Lemma 6.2.1, $S \to S_e^{+v}$ is a twilight extension. By assumption, $(S_e^{+v}, \widehat{\deg}_d^{+v})$ is an extended simplicial surface (no new inner vertices).

Since the definition of $\deg_{S_e^{+v}}$ from Lemma 6.2.1 and the definition of $\widehat{\deg}_d^{+v}$ are inverse, we obtain an extended twilight morphism.

The only vertex not in the image is v. Since $\deg_{S_e^{+v}}(v) = 1$, the twilight morphism is hexagonal if and only if $\widehat{\deg}_d^{+v}(v) = 5$. $\square$

In Lemma 6.2.3, we extended an extended simplicial surface by one face. Assuming the surface we started with was growth–controlled (compare Definition 4.2.4), what can we say about the modified surface? In general, it does not have to be growth–controlled.

Example 6.2.4. *Consider an extended simplicial surface* $(S, \widehat{\deg})$ *such that* ∂S *is a cyclic graph and the external degree–sequence has the form* $(4, 2, 2, 3, 4)$*. Then,* $\widehat{\deg}$ *is growth–controlled.*

We extend it with Lemma 6.2.3 along the edge between the values 3 and 4. This gives the modified sequence $(4, 2, 2, 2, 2, 5, 3)$*, which is not growth–controlled.*

In Example 6.2.4, the issue is the subsequence $(2, 2)$ on one side of the modified edge. If we ban occurrences like this, we can guarantee growth–control.

Lemma 6.2.5. *Let* $(S, \widehat{\deg})$ *be a growth–controlled, extended simplicial surface with a boundary edge* e *such that* $\widehat{\deg}(w) > 2$ *for all* $w \in \eta(e)$*. Additionally, assume that all cyclic intervals* I *containing one* $w \in \eta(e)$ *satisfy* $d_{\widehat{\deg}}(I) \leq 1$*.*

Then, $(S_e^{+v}, \widehat{\deg}_5^{+v})$ *is growth–controlled.*

Proof. We compare the cyclic $\mathbb{N}$–sequences $\widehat{\deg}$ and $\widehat{\deg}_5^{+v}$ in the cyclic graphs ∂S and ∂S_e^{+v}.

Let $\eta(e) = \{w_1, w_2\}$. In order to apply Lemma 3.4.13, we split $\partial S = (V_1, E_1, \eta_1)$ into the cyclic intervals induced by

$$I_1 := \{w_1, w_2\} \qquad\qquad J_1 := V_1 \backslash I_1.$$

We split $\partial S_e^{+v} = (V_2, E_2, \eta_2)$ into

$$I_2 := \{w_1, v, w_2\} \qquad\qquad J_2 := V_2 \backslash I_2.$$

By construction of S_e^{+v}, we have $J_1 = J_2$ and $\widehat{\deg}(w) = \widehat{\deg}_5^{+v}(w)$ for all $w \in J_1$. Thus, Lemma 3.4.13 is applicable.

To prove that $\widehat{\deg}_5^{+v}$ is growth–controlled, we need to check the properties from Definition 3.4.11.

- We have

$$\begin{aligned}
d_{\widehat{\deg}_5^{+v}}(I_2) &= (3 - \widehat{\deg}_5^{+v}(w_1)) + (3 - \widehat{\deg}_5^{+v}(v)) + (3 - \widehat{\deg}_5^{+v}(w_2)) \\
&= (3 - \widehat{\deg}(w_1) + 1) + (3 - 5) + (3 - \widehat{\deg}(w_2) + 1) \\
&= (3 - \widehat{\deg}(w_1)) + (3 - \widehat{\deg}(w_2)) \\
&= d_{\widehat{\deg}}(I_1),
\end{aligned}$$

  so $d_{\widehat{\deg}_5^{+v}}(V_2) \leq 0$ by Lemma 3.4.13.

- Let X be a cyclic interval in ∂S_e^{+v} such that $X \cap I_2$ is a cyclic subinterval of I_2. We distinguish several cases:
  - If $X \cap I_2 = \{w_1, v, w_2\}$, we have

$$\begin{aligned}
d_{\widehat{\deg}_5^{+v}}(X) &= d_{\widehat{\deg}_5^{+v}}(I_2) + d_{\widehat{\deg}_5^{+v}}(X \cap J_2) \\
&= d_{\widehat{\deg}}(I_1) + d_{\widehat{\deg}}(X \cap J_1) \leq 2,
\end{aligned}$$

    since $(S, \widehat{\deg})$ is growth–controlled.
  - If $X \cap I_2 = \{v, w_k\}$, we choose the corresponding interval $\{w_k\} \subseteq I_1$. This gives

$$\begin{aligned}
d_{\widehat{\deg}_5^{+v}}(X) &= d_{\widehat{\deg}_5^{+v}}(\{v, w_k\}) + d_{\widehat{\deg}_5^{+v}}(X \cap J_2) \\
&= (3 - 5) + (3 - \widehat{\deg}_5^{+v}(w_k)) + d_{\widehat{\deg}}(X \cap J_1) \\
&= -1 + (3 - \widehat{\deg}(w_k)) + d_{\widehat{\deg}}(X \cap J_1) \leq -1 + 2 = 0.
\end{aligned}$$

  - If $X \cap I_2 = \{w_k\}$, we choose the corresponding interval $\{w_k\} \subseteq I_1$. If we remember that the defect–sum of the corresponding cyclic interval is bounded by 1, we obtain

$$\begin{aligned}
d_{\widehat{\deg}_5^{+v}}(X) &= d_{\widehat{\deg}_5^{+v}}(\{w_k\}) + d_{\widehat{\deg}_5^{+v}}(X \cap J_2) \\
&= (3 - \widehat{\deg}_5^{+v}(w_k)) + d_{\widehat{\deg}}(X \cap J_1) \\
&= 1 + (3 - \widehat{\deg}(w_k)) + d_{\widehat{\deg}}(X \cap J_1) \leq 1 + 1 = 2.
\end{aligned}$$

  Thus, the defect–sum of all cyclic interval X, where $X \cap I_2$ is a subinterval of I_2, is bounded by 2.

- The final case (by Lemma 3.4.6) is a cyclic interval in ∂S_e^{+v} that intersects I_2 in $\{w_1, w_2\}$. This interval is generated by $V_2 \backslash \{v\}$. But then,

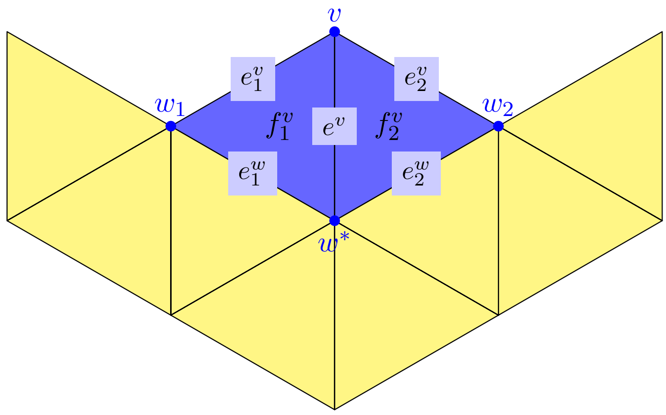
$$d_{\widehat{\deg}_5^{+v}}(V_2 \backslash \{v\}) = d_{\widehat{\deg}_5^{+v}}(V_2) - d_{\widehat{\deg}_5^{+v}}(\{v\}) \leq 0 - (3 - 5) = 2 \leq 2.$$

Since all cyclic intervals are defect–controlled, $\widehat{\deg}_5^{+v}$ is growth–controlled. To show that $(S_e^{+v}, \widehat{\deg}_5^{+v})$ is growth–controlled, we need one more condition according to Definition 4.2.4, namely $\widehat{\deg}_5^{+v}(w) > 1$ for all $w \in V_2$. Consider $\widehat{\deg}_5^{+v}$ from Lemma 6.2.3.

- $\widehat{\deg}_5^{+v}(v) = 5 > 1$.
- For $w \in \eta(e)$, we have $\widehat{\deg}(w) > 2$ by assumption. Thus, $\widehat{\deg}_5^{+v}(w) > 1$.
- For all other vertices $w \in V_1$, we have $\widehat{\deg}_5^{+v}(w) = \widehat{\deg}(w) > 1$, since $(S, \widehat{\deg})$ is growth–controlled. □

6.2.2 Extension by two faces

In this subsection, we consider extensions of simplicial surfaces by two faces. We want to model the scenario illustrated here (where the blue triangles are added):



We also show that this is the unique extension by two faces, such that the vertex w^* becomes an inner vertex

Lemma 6.2.6. *Let $S = (V, E, F, \eta, \varphi)$ be a simplicial surface with boundary vertex $w^* \in V$. There is a unique simplicial surface T with boundary vertex v satisfying*

- $S = T^{-v}$,
- $\deg_T(v) = 2$,
- *w is an inner vertex of T.*

If $e_1^w, e_2^w \in E$ are the boundary edges of S with $\eta(e_k^w) = \{w^*, w_k\}$, the unique T is given as $(V^+, E^+, F^+, \eta^+, \varphi^+)$ with

$$V^+ = V \uplus \{v\}, \qquad E^+ = E \uplus \{e_1^v, e^v, e_2^v\}, \qquad F^+ = F \uplus \{f_1^v, f_2^v\},$$

and

$$\eta^+ : E^+ \to \mathrm{Pot}_2(V^+) \qquad e \mapsto \begin{cases} \eta(e) & e \in E \\ \{v, w^*\} & e = e^v \\ \{v, w_k\} & e = e_k^v \textit{ with } \eta(e_k^w) = \{w^*, w_k\}, \end{cases}$$

$$\varphi^+ : F^+ \to \mathrm{Pot}_3(E^+) \qquad f \mapsto \begin{cases} \varphi(f) & f \in F \\ \{e^v, e_k^w, e_k^v\} & f = f_k^v. \end{cases}$$

We also have

$$\deg_T(w) = \begin{cases} \deg_S(w) & w \notin \{v, w_1, w_2\} \\ \deg_S(w) + 1 & w \in \{w_1, w_2\} \\ \deg_S(w) + 2 & w = w^* \\ 2 & w = v \end{cases}$$

Proof. We start by showing the uniqueness of $T = (V^+, E^+, F^+, \eta^+, \varphi^+)$.

- From $V^+ \backslash \{v\} = V$ we obtain $V^+ = V \uplus \{v\}$.
- From $\deg_T(V) = 2$ we obtain $F^+ = F \uplus \{f_1^v, f_2^v\}$.
- Since v is a boundary vertex, we have a maximal umbrella $(e_1^v, f_1^v, e^v, f_2^v, e_2^v)$ around v. This also gives $E^+ = E \uplus \{e_1^v, e^v, e_2^v\}$.
- Since w is an inner vertex with incident boundary edges e_1^w and e_2^2, we can choose $e_1^w \in \varphi^+(f_1^v)$ without loss of generality. Since $v \notin \eta(e_2^w)$, we conclude $e_2^w \notin \varphi^+(f_1^v)$, implying $e_2^w \in \varphi^+(f_2^v)$.

This also proves the statement about the map $\deg_T$.

Next, we show that $T = (V^+, E^+, F^+, \eta^+, \varphi^+)$ defined as above actually is a simplicial surface. However, this is easy to see (similar to Lemma 6.2.1). ☐

The extension constructed in Lemma 6.2.6 deserves a name.

Definition 6.2.7. *We call the unique simplicial surface from Lemma 6.2.6* $S^{+v}_{e_1^w, e_2^w}$.

Next, we generalise the extension from Lemma 6.2.6 to extended simplicial surfaces, analogous to Lemma 6.1.8. However, there is some ambiguity: Since the vertex v is "new", we have no information about its external degree. To determine it uniquely, we need further restrictions. In our case, the construction in Chapter 8 requires the extended polygonal twilight morphisms to be *hexagonal* (compare Definition 4.2.11). This determines the external degree of v uniquely.

We also change our notation. In Lemma 6.2.3, it was sufficient to talk about a single edge with adjacent vertices. While this description is very simple, it is not suitable for more complicated assumptions concerning the boundary. Instead, we talk about vertex–edge–paths on the boundary.

Lemma 6.2.8. *Let $(S, \widehat{\deg})$ be an extended simplicial surface and $(v_0, e_1, v_1, e_2, v_2)$ a vertex–edge–path on the boundary, with*

$$\widehat{\deg}(v_0) > 1, \qquad \widehat{\deg}(v_1) = 2, \qquad \widehat{\deg}(v_2) > 1.$$

Then, $(S^{+v}_{e_1,e_2}, \widehat{\deg}^{+v}_d)$ (with $d \in \mathbb{Z}_{\geq 1}$) is an extended simplicial surface with

$$\widehat{\deg}^{+v}_d(w) = \begin{cases} \widehat{\deg}(w) & w \notin \{v, v_0, v_1, v_2\} \\ \widehat{\deg}(w) - 1 & w \in \{v_0, v_2\} \\ 0 & w = v_1 \\ d & w = v. \end{cases}$$

Furthermore, $(S, \widehat{\deg}) \to (S^{+v}_{e_1,e_2}, \widehat{\deg}^{+v}_d)$ is an extended twilight morphism. It is hexagonal if and only if $d = 4$.

Proof. By Lemma 6.2.6, $S \to S^{+v}_{e_1,e_2}$ is a twilight morphism. By assumption, $(S^{+v}_e, \widehat{\deg}^{+v}_d)$ is an extended simplicial surface (only v_1 becomes an inner vertex).

To show that $(S, \widehat{\deg}) \to (S^{+v}_{e_1,e_2}, \widehat{\deg}^{+v}_d)$ is an extended twilight morphism, we have to show $\deg(w) + \widehat{\deg}(w) = \deg_{S^{+v}_{e_1,e_2}} + \widehat{\deg}^{+v}_d(w)$ for all vertices w in S.

- If $w \notin \{v_0, v_1, v_2\}$, this is clear (since the degrees of the extension do not change).
- If $w \in \{v_0, v_2\}$, the definitions are inverse to each other.
- For $w = v_1$, the assumption $\widehat{\deg}(v_1) = 2$ is crucial.

The only vertex not in the image is v. Since $\deg_{S^{+v}_{e_1,e_2}}(v) = 2$, the twilight morphism is hexagonal if and only if $\widehat{\deg}^{+v}_d(v) = 4$. □

In Lemma 6.2.8, we extended an extended simplicial surface by two faces. Assuming the surface we started with was growth–controlled (compare Definition 4.2.4), what can we say about the modified surface? In contrast to extending by one face (compare Example 6.2.4), the modified surface is almost always growth–controlled (only very mild assumptions are necessary).

Lemma 6.2.9. *Let $(S, \widehat{\deg})$ be a growth–controlled, extended simplicial surface with boundary vertex–edge–path $(v_0, e_1, v_1, e_2, v_2)$ satisfying*

$$\widehat{\deg}(v_0) > 2, \qquad \widehat{\deg}(v_1) = 2, \qquad \widehat{\deg}(v_2) > 2.$$

Then, $(S^{+v}_{e_1,e_2}, \widehat{\deg}^{+v}_4)$ is growth–controlled.

Proof. We compare the cyclic $\mathbb{N}$–sequences $\widehat{\deg}$ and $\widehat{\deg}^{+v}_4$ in the cyclic graphs ∂S and $\partial S^{+v}_{e_1,e_2}$.

We would like to apply Lemma 3.4.13. To do so, we split $\partial S = (V_1, E_1, \eta_1)$ into the cyclic intervals induced by

$$I_1 := \{v_0, v_1, v_2\} \qquad\qquad J_1 := V_1 \backslash I_1.$$

We split $\partial S^{+v}_{e_1,e_2} = (V_2, E_2, \eta_2)$ into

$$I_2 := \{v_0, v, v_2\} \qquad\qquad J_2 := V_2 \backslash I_2.$$

By construction of $S^{+v}_{e_1,e_2}$, we have $J_1 = J_2$ and $\widehat{\deg}(w) = \widehat{\deg}_4^{+v}(w)$ for all $w \in J_1$. Thus, Lemma 3.4.13 is applicable.

To prove that $\widehat{\deg}_4^{+v}$ is growth–controlled, we need to check the properties from Definition 3.4.11. We repeatedly make use of the following facts:

$$\begin{aligned}
\widehat{\deg}_4^{+v}(v_0) &= \widehat{\deg}(v_0) - 1, \\
\widehat{\deg}_4^{+v}(v) &= 4, \\
\widehat{\deg}_4^{+v}(v_2) &= \widehat{\deg}(v_2) - 1,
\end{aligned}$$

- We have

$$\begin{aligned}
d_{\widehat{\deg}_4^{+v}}(I_2) &= (3 - \widehat{\deg}_4^{+v}(v_0)) + (3 - \widehat{\deg}_4^{+v}(v)) + (3 - \widehat{\deg}_4^{+v}(v_2)) \\
&= (3 - \widehat{\deg}(v_0) + 1) + (3 - 4) + (3 - \widehat{\deg}(v_2) + 1) \\
&= (3 - \widehat{\deg}(v_0)) + (3 - 2) + (3 - \widehat{\deg}(v_2)) \\
&= (3 - \widehat{\operatorname{def}}(v_0)) + (3 - \widehat{\deg}(v_1)) + (3 - \widehat{\deg}(v_2)) \\
&= d_{\widehat{\deg}}(I_1),
\end{aligned}$$

  so $d_{\widehat{\deg}_4^{+v}}(V_2) \leq 0$ by Lemma 3.4.13.

- Consider a cyclic interval C_2 contained in I_2. We have to find a matching cyclic interval C_1 in I_1 with $d_{\widehat{\deg}}(C_1) \geq d_{\widehat{\deg}_4^{+v}}(C_2)$. The cases are as follows:
  - If $C_2 = \{v_0\}$, we choose $C_1 = \{v_0, v_1\}$ to obtain

$$d_{\widehat{\deg}_4^{+v}}(C_2) = 3 - \widehat{\deg}_4^{+v}(v_0) = 3 - \widehat{\deg}(v_0) + (-1) = d_{\widehat{\deg}}(C_1).$$

  - If $C_2 = \{v_2\}$, we choose $C_1 = \{v_1, v_2\}$ to obtain

$$d_{\widehat{\deg}_4^{+v}}(C_2) = 3 - \widehat{\deg}_4^{+v}(v_2) = 3 - \widehat{\deg}(v_2) + (-1) = d_{\widehat{\deg}}(C_1).$$

  - If $C_2 = \{v_0, v\}$, we choose $C_1 = \{v_0\}$ to obtain

$$d_{\widehat{\deg}_4^{+v}}(C_2) = 3 - \widehat{\deg}_4^{+v}(v_0) + 1 = 3 - \widehat{\deg}(v_0) = d_{\widehat{\deg}}(C_1).$$

– If $C_2 = \{v, v_2\}$, we choose $C_1 = \{v_2\}$ to obtain

$$d_{\widehat{\deg}_4^{+v}}(C_2) = 3 - \widehat{\deg}_4^{+v}(v_2) + 1 = 3 - \widehat{\deg}(v_2) = d_{\widehat{\deg}}(C_1).$$

– $C_2 = \{v, v_0, v_2\}$, we choose $C_1 = \{v_0, v_1, v_2\}$ to obtain

$$\begin{aligned} d_{\widehat{\deg}}(C_2) &= 3 - \widehat{\deg}_4^{+v}(v_0) + 1 + 3 - \widehat{\deg}_4^{+v_2} \\ &= 3 - \widehat{\deg}(v_0) + (-1) + 3 - \widehat{\deg}(v_2) \\ &= d_{\widehat{\deg}}(C_1). \end{aligned}$$

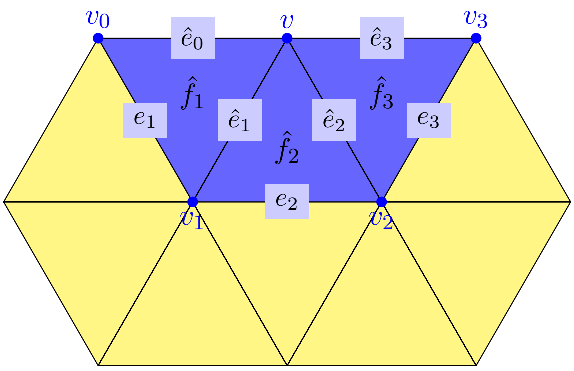
- The final case (by Lemma 3.4.6) is a cyclic interval in $\partial S^{+v}_{e_1,e_2}$ that intersects I_2 in $\{v_0, v_2\}$. This interval is generated by $V_2\backslash\{v\}$. But then,

$$d_{\widehat{\deg}_4^{+v}}(V_2\backslash\{v\}) = d_{\widehat{\deg}_4^{+v}}(V_2) - d_{\widehat{\deg}_4^{+v}}(\{v\}) \leq 0 - (3-4) = 1 \leq 2.$$

Since all cyclic intervals are defect–controlled, $\widehat{\deg}_4^{+v}$ is growth–controlled. To show that $(S^{+v}_{e_1,e_2}, \widehat{\deg}_4^{+v})$ is growth–controlled, we need one more condition according to Definition 4.2.4, namely $\widehat{\deg}_4^{+v}(w) > 1$ for all $w \in V_2$. This is guaranteed by our analysis from the start of the proof. $\square$

6.2.3 Extension by three faces

In this subection, we consider extensions of simplicial surfaces by three faces. We want to model the scenario illustrated here (where the blue triangles are added):



We also show that this is the unique extension by three faces, such that both vertices v_1 and v_2 become inner vertices.

Lemma 6.2.10. *Let $S = (V, E, F, \eta, \varphi)$ be a simplicial surface with non–repeating boundary vertex–edge–path $(v_0, e_1, v_1, e_2, v_2, e_3, v_3)$, where $v_0 \neq v_3$. There is a unique simplicial surface T with boundary vertex v satisfying*

- $S = T^{-v}$,
- $\deg_T(v) = 3$,
- v_1 *and* v_2 *are inner vertices of* T.

The unique T *is given as* $(V^+, E^+, F^+, \eta^+, \varphi^+)$ *with*

$$V^+ = V \uplus \{v\}, \qquad E^+ = E \uplus \{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\}, \qquad F^+ = F \uplus \{\hat{f}_1, \hat{f}_2, \hat{f}_3\},$$

and

$$\eta^+ : E^+ \to \mathrm{Pot}_2(V^+) \qquad e \mapsto \begin{cases} \eta(e) & e \in E \\ \{v, v_k\} & e = \hat{e}_k. \end{cases}$$

$$\varphi^+ : F^+ \to \mathrm{Pot}_3(E^+) \qquad f \mapsto \begin{cases} \varphi(f) & f \in F \\ \{e_k, \hat{e}_{k-1}, \hat{e}_k\} & f = \hat{f}_k. \end{cases}$$

We also have

$$\deg_T(w) = \begin{cases} \deg_S(w) & w \notin \{v, v_0, v_1, v_2, v_3\} \\ \deg_S(w) + 1 & w \in \{v_0, v_3\} \\ \deg_S(w) + 2 & w \in \{v_1, v_2\} \\ 3 & w = v. \end{cases}$$

Proof. We start by showing the uniqueness of $T = (V^+, E^+, F^+, \eta^+, \varphi^+)$.

- From $V^+ \backslash \{v\} = V$ we obtain $V^+ = V \uplus \{v\}$.
- From $\deg_T(v) = 3$ we obtain $F^+ = F \uplus \{\hat{f}_1, \hat{f}_2, \hat{f}_3\}$.
- If two edges from $\{e_1, e_2, e_3\}$ were incident to the same $\hat{f}_k$, the vertex v could not be incident to that face. Since all e_k are inner vertices in T (they are adjacent to the inner vertices v_1 or v_2), we have (without loss of generality), $e_k \in \varphi^+(\hat{f}_k)$.
- $e_k \in \varphi^+(\hat{f}_k)$ implies $(\eta^+ \between \varphi^+)(\hat{f}_k) = \{v_{k-1}, v_k, v\}$. Thus, there are edges $\hat{e}_0, \ldots, \hat{e}_3$ with $\eta^+(\hat{e}_k) = \{v, v_k\}$. From $v_1 \neq v_3$ we can deduce that $\hat{e}_0 \neq \hat{e}_3$.

This also proves the statement about the map $\deg_T$.
It is easy to see that this $T = (V^+, E^+, F^+, \eta^+, \varphi^+)$ is a simplicial surface. ☐

The extension constructed in Lemma 6.2.10 deserves a name.

Definition 6.2.11. *We call the unique simplicial surface from Lemma 6.2.10* $S^{+v}_{e_1,e_2,e_3}$.

Next, we generalise the extension from Lemma 6.2.10 to extended simplicial surfaces, analogous to Lemma 6.1.8. However, there is some ambiguity: Since the vertex v is "new", we have no information about its external degree. To determine it uniquely, we need further restrictions. In our case, the construction in Chapter 8 requires the extended polygonal twilight morphisms to be *hexagonal* (compare Definition 4.2.11). This determines the external degree of v uniquely.

Lemma 6.2.12. *Let* $(S, \widehat{\deg})$ *be an extended simplicial surface with a non–repeating vertex–edge–path* $(v_0, e_1, v_1, e_2, v_2, e_3, v_3)$ *on the boundary, such that* $v_0 \neq v_3$ *and*

$$\widehat{\deg}(v_0) > 1, \qquad \widehat{\deg}(v_1) = 2, \qquad \widehat{\deg}(v_2) = 2, \qquad \widehat{\deg}(v_3) > 1.$$

Then, $(S^{+v}_{e_1,e_2,e_3}, \widehat{\deg}^{+v}_d)$ *is an extended simplicial surface with*

$$\widehat{\deg}^{+v}_d(w) = \begin{cases} \widehat{\deg}(w) & w \notin \{v, v_0, v_1, v_2, v_3\} \\ \widehat{\deg}(w) - 1 & w \in \{v_0, v_3\} \\ 0 & w \in \{v_1, v_2\} \\ d & w = v. \end{cases}$$

Furthermore, $(S, \widehat{\deg}) \to (S^{+v}_{e_1,e_2,e_3}, \widehat{\deg}^{+v}_d)$ *is an extended twilight morphism. It is hexagonal if and only if* $d = 3$.

Proof. By Lemma 6.2.10, the inclusion $S \to S^{+v}_{e_1,e_2,e_3}$ is a twilight morphism. By assumption, $(S^{+v}_{e_1,e_2,e_3}, \widehat{\deg}^{+v}_d)$ is an extended simplicial surface (only v_1 and v_2 become inner vertices).

To show that $(S, \widehat{\deg}) \to (S^{+v}_{e_1,e_2,e_3}, \widehat{\deg}^{+v}_d)$ is an extended twilight morphism, we have to show $\deg(w) + \widehat{\deg}(w) = \deg_{S^{+v}_{e_1,e_2,e_3}} + \widehat{\deg}^{+v}_d(w)$ for all vertices w in S.

- If $w \notin \{v_0, v_1, v_2, v_3\}$, this is clear (since the degrees of the extension do not change).
- If $w \in \{v_0, v_3\}$, the definitions are inverse to each other.
- For $w \in \{v_1, v_2\}$, the assumptions $\widehat{\deg}(v_1) = \widehat{\deg}(v_2) = 2$ are crucial.

The only vertex not in the image is v. Since $\deg_{S^{+v}_{e_1,e_2,e_3}}(v) = 3$, the twilight morphism is hexagonal if and only if $\widehat{\deg}^{+v}_d(v) = 3$. $\square$

In Lemma 6.2.12, we extended an extended simplicial surface by three faces. If the surface we started with was growth–controlled (compare Definition 4.2.4), the modified surface is growth–controlled as well. Contrast this with the extension by one face in Lemma 6.2.5, where we need very strict assumptions (restrictions on possible defect–sums), and the extension by two faces in Lemma 6.2.9, where some mild assumptions (restrictions on possible external degrees) are necessary.

Lemma 6.2.13. *Let* $(S, \widehat{\deg})$ *be a growth–controlled extended simplicial surface with boundary vertex–edge–path* $(v_0, e_1, v_1, e_2, v_2, e_3, v_3)$ *with* $v_0 \neq v_3$ *and*

$$\widehat{\deg}(v_1) = \widehat{\deg}(v_2) = 2.$$

Then, $(S^{+v}_{e_1,e_2,e_3}, \widehat{\deg}^{+v}_3)$ *is growth–controlled.*

Proof. First, we have to show that $(S^{+v}_{e_1,e_2,e_3}, \widehat{\deg}_3^{+v})$ actually exists. By Lemma 6.2.12, we need $\widehat{\deg}(v_0) > 1$ and $\widehat{\deg}(v_3) > 1$. Since $\widehat{\deg}$ is defect–controlled, we have

$$2 \geq d_{\widehat{\deg}}(\{v_0, v_1, v_2\}) = 3 - \widehat{\deg}(v_0) + 2 = 5 - \widehat{\deg}(v_0).$$

so $\widehat{\deg}(v_0) \geq 3$. The same argument applies to v_3.

To show that $\widehat{\deg}_3^{+v}$ is growth–controlled, we compare the cyclic graphs ∂S and $\partial S^{+v}_{e_1,e_2,e_3}$, together with the cyclic $\mathbb{N}$–sequences $\widehat{\deg}$ and $\widehat{\deg}_3^{+v}$.

We would like to apply Lemma 3.4.13. To do so, we split $\partial S = (V_1, E_1, \eta_1)$ into the cyclic intervals induced by

$$I_1 := \{v_0, v_1, v_2, v_3\} \qquad\qquad J_1 := V_1 \backslash I_1.$$

We split $\partial S^{+v}_{e_1,e_2,e_3} = (V_2, E_2, \eta_2)$ into

$$I_2 := \{v_0, v, v_3\} \qquad\qquad J_2 := V_2 \backslash I_2.$$

By construction of $S^{+v}_{e_1,e_2,e_3}$, we have $J_1 = J_2$ and $\widehat{\deg}(w) = \widehat{\deg}_3^{+v}(w)$ for all $w \in J_1$. Thus, Lemma 3.4.13 is applicable.

To prove that $\widehat{\deg}_3^{+v}$ is growth–controlled, we check the conditions of Definition 3.4.11.

- We have

$$\begin{aligned}
d_{\widehat{\deg}_3^{+v}}(I_2) &= (3 - \widehat{\deg}_3^{+v}(v_0)) + (3 - \widehat{\deg}_3^{+v}(v)) + (3 - \widehat{\deg}_3^{+v}(v_3)) \\
&= (3 - \widehat{\deg}(v_0) + 1) + (3 - 3) + (3 - \widehat{\deg}(v_2) + 1) \\
&= (3 - \widehat{\deg}(v_0)) + (3 - 2) + (3 - 2) + (3 - \widehat{\deg}(v_3)) \\
&= (3 - \widehat{\mathrm{def}}(v_0)) + (3 - \widehat{\deg}(v_1)) + (3 - \widehat{\deg}(v_2)) + (3 - \widehat{\deg}(v_3)) \\
&= d_{\widehat{\deg}}(I_1),
\end{aligned}$$

  so $d_{\widehat{\deg}_3^{+v}}(V_2) \leq 0$ by Lemma 3.4.13.

- Consider a cyclic interval C_2 contained in I_2. We have to find a matching cyclic interval C_1 in I_1 with $d_{\widehat{\deg}}(C_1) \geq d_{\widehat{\deg}_3^{+v}}(C_2)$. We choose as follows:
  - If $C_2 = \{v_0\}$, we choose $C_1 = \{v_0, v_1\}$ to obtain

$$d_{\widehat{\deg}_3^{+v}}(C_2) = 3 - \widehat{\deg}_3^{+v}(v_0) = 3 - \widehat{\deg}(v_0) + 1 = d_{\widehat{\deg}}(C_1).$$

  - If $C_2 = \{v_3\}$, we choose $C_1 = \{v_2, v_3\}$ to obtain

$$d_{\widehat{\deg}_3^{+v}}(C_2) = 3 - \widehat{\deg}_3^{+v}(v_3) = 3 - \widehat{\deg}(v_3) + 1 = d_{\widehat{\deg}}(C_1).$$

– If $C_2 = \{v_0, v\}$, we choose $C_1 = \{v_0, v_1\}$ to obtain

$$d_{\widehat{\deg}_3^{+v}}(C_2) = 3 - \widehat{\deg}_3^{+v}(v_0) = 3 - \widehat{\deg}(v_0) + 1 = d_{\widehat{\deg}}(C_1).$$

– If $C_2 = \{v_3, v\}$, we choose $C_1 = \{v_2, v_3\}$ to obtain

$$d_{\widehat{\deg}_3^{+v}}(C_2) = 3 - \widehat{\deg}_3^{+v}(v_3) = 3 - \widehat{\deg}(v_3) + 1 = d_{\widehat{\deg}}(C_1).$$

– If $C_2 = \{v_0, v_3, v\}$, we choose $C_1 = \{v_0, v_1, v_2, v_3\}$ to obtain

$$\begin{aligned} d_{\widehat{\deg}_3^{+v}}(C_2) &= 3 - \widehat{\deg}_3^{+v}(v_0) + 3 - \widehat{\deg}_3^{+v}(v_3) \\ &= 3 - \widehat{\deg}(v_0) + 1 + 1 + 3 - \widehat{\deg}(v_3) \\ &= d_{\widehat{\deg}}(C_1). \end{aligned}$$

- The final case (by Lemma 3.4.6) is a cyclic interval in $\partial S^{+v}_{e_1,e_2,e_3}$ that intersects I_2 in $\{v_0, v_3\}$. This interval is generated by $V_2 \backslash \{v\}$. But then,

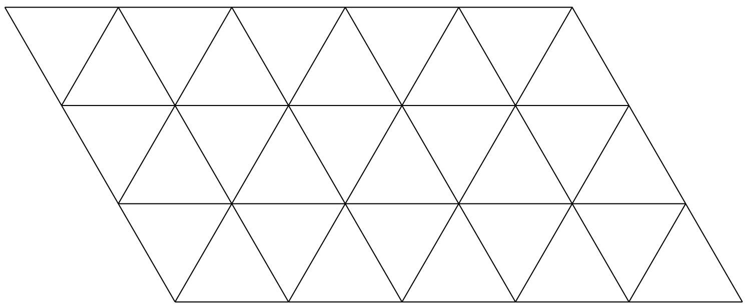
$$d_{\widehat{\deg}_3^{+v}}(V_2 \backslash \{v\}) = d_{\widehat{\deg}_3^{+v}}(V_2) - d_{\widehat{\deg}_3^{+v}}(\{v\}) \leq 0 - (3 - 4) = 1 \leq 2.$$

Since all cyclic intervals are defect–controlled, $\widehat{\deg}_3^{+v}$ is growth–controlled. To show that $(S^{+v}_{e_1,e_2,e_3}, \widehat{\deg}_3^{+v})$ is growth–controlled, we need one more condition according to Definition 4.2.4, namely $\widehat{\deg}_3^{+v}(w) > 1$ for all $w \in V_2$. This is guaranteed by our assumption on $\widehat{\deg}$ (compare the definition of $\widehat{\deg}_3^{+v}$ in Lemma 6.2.12). □

7 Hexagonal lattice

While working with combinatorial objects, one usually eschews their geometric realisations in favour of notions like incidence or colouring. But sometimes, it is easiest to define (and work with) a combinatorial structure by embedding it into a certain space.

In this chapter, we work with the infinite hexagonal lattice, which we visualise like an infinitely continued version of this illustration:



In Section 7.1, we construct this lattice geometrically and show that it can be represented as a combinatorial surface. Then, we construct its automorphism group explicitly.

In Section 7.2, we work with paths within the hexagonal lattice. We pay particular attention to those combinatorial paths that correspond to `Jordan`–curves in the embedded lattice, and give a criterion to construct such a path with certain specifications.

In Section 7.3, we follow a differential–geometric approach and represent each twisted triangular surface as a subset of the hexagonal lattice, together with certain "transition maps". We show how the combinatorial properties of the twisted triangular surface relate to group–theoretic properties of the transition maps.

7.1 Definition and basic properties

In this section, we define the infinite hexagonal lattice explicitly, and construct its automorphism group. To facilitate this, we define some helpful notation.

Notation 7.1.1. *If $p \in \mathbb{Z}^2$, we define*

$$p^{+0} := p + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad p^{++} := p + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad p^{0+} := p + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$p^{-0} := p - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad p^{--} := p - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad p^{0-} := p - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Definition 7.1.2. *The **hexagonal lattice surface** is the vertex–faithful simplicial surface* (V, E, F, η, φ) *with*

- $V = \mathbb{Z}^2$,
- $E = E_- \uplus E_\backslash \uplus E_/$ *with*

$$\begin{aligned} E_- &= \{\{p, p^{+0}\} \mid p \in V\}, \\ E_\backslash &= \{\{p, p^{++}\} \mid p \in V\}, \\ E_/ &= \{\{p, p^{0+}\} \mid p \in V\} \end{aligned}$$

- $F = F_+ \uplus F_-$ *with*

$$\begin{aligned} F_+ &= \{\{p, p^{+0}, p^{++}\} \mid p \in V\} \\ F_- &= \{\{p, p^{++}, p^{0+}\} \mid p \in V\} \end{aligned}$$

- $\eta : E \to \mathrm{Pot}_2(V)$ *is the identity mapping,*
- $\varphi : F \to \mathrm{Pot}_3(E)$ *with* $\varphi(f) = \mathrm{Pot}_2(f)$.

*The **hexagonal lattice embedding** is the map*

$$V \to \mathbb{R}^2, \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}.$$

*The set of images of this map is the **hexagonal lattice**.*

Well–defined. We have to show that the hexagonal lattice surface is in fact a simplicial surface. We start with checking the conditions for a vertex–faithful triangular complex from Lemma 2.7.5:

1. The maps η and $id : F \to \mathrm{Pot}_3(V)$ are injective.
2. The vertex p is contained in the edge $\{p, p^{++}\}$.
3. The edges $\{p, p^{+0}\}$ and $\{p, p^{++}\}$ are contained in the face $\{p, p^{+0}, p^{++}\}$. The edge $\{p, p^{0+}\}$ is contained in the face $\{p, p^{++}, p^{0+}\}$.
4. For the face $\{p, p^{+0}, p^{++}\}$, only the subset $\{p^{+0}, p^{++}\}$ has to be checked. With $q := p^{+0}$ we have $\{p^{+0}, p^{++}\} = \{q, q^{0+}\}$.

   For the face $\{p, p^{++}, p^{0+}\}$, only the subset $\{p^{++}, p^{0+}\}$ has to be checked. With $q := p^{0+}$ we have $\{p^{++}, p^{0+}\} = \{q^{+0}, q\}$.

Therefore, (V, E, F, η, φ) is a vertex–faithful triangular complex.

To show that it is a simplicial surface, we have to show that there are neither ramified edges nor ramified vertices (compare Definition 2.5.27).

- Every edge has a "direction", one of $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\1\end{pmatrix}$. In each face, there is one edge of each direction. For both types of faces, if we know one edge specifically, the face is unique. Thus, every edge is incident to exactly two faces (one of each type).

- Finally, consider a vertex p. We want to show that this is an inner vertex. To show this, we compute all edges and faces incident to this vertex. The edges are

$$\{\{p, p^{+0}\}, \{p, p^{++}\}, \{p, p^{0+}\}, \{p^{-0}, p\}, \{p^{--}, p\}, \{p^{0-}, p\}\}$$

  and the faces are

$$\begin{aligned}\{\{p, p^{+0}, p^{++}\}, \{p, p^{++}, p^{0+}\}, \{p^{-0}, p, p^{0+}\},\\ \{p^{--}, p, p^{-0}\}, \{p^{--}, p^{0-}, p\}, \{p^{0-}, p^{+0}, p\}\}.\end{aligned}$$

  It is easy to see that we can arrange these edges and faces in one umbrella. Thus, v is an inner vertex.

This completes the proof. $\square$

Next, we consider the automorphisms of the hexagonal lattice surface. We start by relating its action on the flags of H to its action on vertex triples.

Remark 7.1.3. *Let $H = (V, E, F, \eta, \varphi)$ be the hexagonal lattice surface. The action of* $\mathrm{Aut}(H)$ *on the flags of H is equivariant to the action of* $\mathrm{Aut}(H)$ *on triples*

$$\{(v_1, v_2, v_3) \in V^3 \mid \exists f \in F \textit{ with } \eta \between \varphi(f) = \{v_1, v_2, v_3\}\}.$$

Proof. Let $\mathcal{F}(H)$ refer to the flags of H. We define the map

$$\rho : \mathcal{F}(H) \to \{(v_1, v_2, v_3) \in V^3 \mid \exists f \in F \text{ with } \eta \between \varphi(f) = \{v_1, v_2, v_3\}\}$$

as follows: The flag $(v, e, f) \in \mathcal{F}(H)$ is mapped to the triple (v, v_e, v_f) with $\eta(e) = \{v, v_e\}$ and $(\eta \between \varphi)(f) = \{v, v_e, v_f\}$.

We have to show that ρ is bijective. Since H is vertex–faithful, there is at most one preimage of each $x \in V^3$. By definition, there is always at least one preimage.

Finally, we show equivariance (Definition 4.3.4). Let $\mu = (\mu_V, \mu_E, \mu_F) \in \mathrm{Aut}(H)$. Then, $\mu(v, e, f) = (\mu_V(v), \mu_E(e), \mu_F(f))$. Since μ is an automorphism, we have

$$\eta(\mu_E(e)) = \{\mu_V(v), \mu_V(v_e)\}, \qquad (\eta \between \varphi)(\mu_F(f)) = \{\mu_V(v), \mu_V(v_e), \mu_V(v_f)\}.$$

This implies the equivariance of the actions. $\square$

The rewriting of Remark 7.1.3 makes it easy to concretely write down the automorphism group.

Lemma 7.1.4. *Let H be the hexagonal lattice surface.* $\mathrm{Aut}(H) = D_{12} \ltimes \mathbb{Z}^2$*, with* $D_{12} = \langle r, m \rangle$ *and*

$$r = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad m = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

with action

$$\mathrm{Aut}(H) \times \mathbb{Z}^2 \to \mathbb{Z}^2 \qquad\qquad ((M,t), \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}) \mapsto M \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t.$$

Furthermore, $\mathrm{Aut}(H)$ *acts regularly on the flags of H.*

Proof. The element $(M,t) \in D_{12} \ltimes \mathbb{Z}^2$ maps $x \in \mathbb{Z}^2$ to $Mx + t$. To show that this map induces an automorphism of H, it suffices to show that its generators induce automorphisms of H.

- Clearly, translations by $t \in \mathbb{Z}^2$ preserve edges and faces.
- For the matrix r, we have

  $$r.p^{+0} = (rp)^{++}, \qquad r.p^{++} = (rp)^{0+}, \qquad r.p^{0+} = (rp)^{-0}.$$

  This induces the following map on edges:

  $$\begin{aligned} \{p, p^{+0}\} &\mapsto \{rp, (rp)^{++}\} \\ \{p, p^{++}\} &\mapsto \{rp, (rp)^{0+}\} \\ \{p, p^{0+}\} &\mapsto \{(rp)^{-0}, rp\} \end{aligned}$$

  On the faces, it acts like this:

  $$\begin{aligned} \{p, p^{+0}, p^{++}\} &\mapsto \{rp, (rp)^{++}, (rp)^{0+}\} \\ \{p, p^{++}, p^{0+}\} &\mapsto \{(rp)^{-0}, rp, (rp)^{0+}\} \end{aligned}$$

- For the matrix m, we have

  $$m.p^{+0} = (mp)^{0-}, \qquad m.p^{++} = (mp)^{--}, \qquad m.p^{0+} = (mp)^{-0}.$$

  This induces the following map on edges:

  $$\begin{aligned} \{p, p^{+0}\} &\mapsto \{(mp)^{0-}, mp\} \\ \{p, p^{++}\} &\mapsto \{(mp)^{--}, mp\} \\ \{p, p^{0+}\} &\mapsto \{(mp)^{-0}, mp\} \end{aligned}$$

  On the faces, it acts like this:

  $$\begin{aligned} \{p, p^{+0}, p^{++}\} &\mapsto \{(mp)^{--}, (mp)^{0-}, mp\} \\ \{p, p^{++}, p^{0+}\} &\mapsto \{(mp)^{--}, mp, (mp)^{-0}\} \end{aligned}$$

Thus, $D_{12} \ltimes \mathbb{Z}^2 \leq \mathrm{Aut}(H)$. We consider the action on the flags of H, under the equivariance of Remark 7.1.3.

Let $(v_1, v_2, v_3) \in V^3$ and $(w_1, w_2, w_3) \in V^3$. By applying translations, we can assume that $v_1 = w_1 = 0$. Since $\langle r \rangle$ acts transitively on

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\},$$

we can furthermore assume that $v_2 = w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then, the final vertex lies in $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$. Since

$$mr^2. \begin{pmatrix} 1 \\ 0 \end{pmatrix} = m \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the group $D_{12} \ltimes \mathbb{Z}^2$ acts transitively on the flags of H.

For any $\alpha \in \mathrm{Aut}(H)$, there is a $\beta \in D_{12} \ltimes \mathbb{Z}^2$ such that $\alpha\beta$ stabilises a flag (v_1, v_2, v_3). But if the images of $\{v_1, v_2, v_3\}$ are determined, the images of their neighbours are as well. Therefore, the automorphism stabilises H pointwise. This implies $\alpha \in D_{12} \ltimes \mathbb{Z}^2$. Since the stabiliser of each flag is trivial, the regularity of the action also follows. $\square$

For concrete calculations, it is useful to know that we can write the elements of $D_{12} = \langle r, m \rangle$ uniquely.

Remark 7.1.5. *Let $g \in D_{12} = \langle r, m \rangle$ with notation from Lemma 7.1.4. Then, $g = r^k$ or $g = r^k m$ with $0 \leq k \leq 5$.*

Proof. We can write any $g \in \langle r, m \rangle$ as a word in r and m, since both have finite order. Since $mr = r^5 m$, the claim follows. $\square$

7.2 Polygons

In the previous Section 7.1, we defined the hexagonal lattice surface. In this section, we deal with certain vertex–edge–paths (compare Definition 5.2.10) in the hexagonal lattice surface. Namely, those paths that correspond to `Jordan`–curves in the hexagonal lattice embedding.

This allows us to define an *inner degree* for those curves. The main result of this section is Lemma 7.2.7, in which we construct one of these paths with certain specified inner degrees.

We start with the definition of these paths. Since the hexagonal lattice surface is vertex–faithful, the path is uniquely determined by its vertices.

Definition 7.2.1. *Let $H = (V, E, F, \eta, \varphi)$ be the hexagonal lattice surface. A* ***hexagonal path*** *is a sequence $(p_1, p_2, \ldots, p_n)$ with $p_i \in \mathbb{Z}^2$ such that p_i and p_{i+1} are adjacent in H.*

*It is called **closed** if $p_1 = p_n$ and **non–intersecting** if $p_i \neq p_j$ for $i \neq j$ (exception for the pair $\{1, n\}$).*

*A closed, non–intersecting hexagonal path is called **hexagonal polygon path**.*

A particular simple hexagonal polygon path consists of all points with fixed distance d to the origin (where we measure distance as the number of edges in a minimal vertex–edge–paths).

Example 7.2.2. *Let H be the hexagonal lattice surface and $d \geq 1$. Then,*

$$(p_0, p_1, \ldots, p_{6d})$$

with

$$p_k = \begin{cases} \begin{pmatrix} d \\ k \end{pmatrix} & 0 \leq k \leq d \\ \begin{pmatrix} 2d - k \\ d \end{pmatrix} & d \leq k \leq 2d \\ \begin{pmatrix} 2d - k \\ 3d - k \end{pmatrix} & 2d \leq k \leq 3d \\ \begin{pmatrix} -d \\ 3d - k \end{pmatrix} & 3d \leq k \leq 4d \\ \begin{pmatrix} k - 5d \\ -d \end{pmatrix} & 4d \leq k \leq 5d \\ \begin{pmatrix} k - 5d \\ k - 6d \end{pmatrix} & 5d \leq k \leq 6d \end{cases}$$

is a hexagonal polygon path.

Proof. It is easy to see that the definition of p_k is self–consistent and that it defines a closed hexagonal path. It remains to show that it is non–intersecting. Assume that $p_k = p_l$ for some $k \neq l$. Since all points within one case of the definition are distinct, k and l belong to different cases.

If we consider the first component, it is clear that the cases for $0 \leq k \leq d$ and $3d \leq k \leq 4d$ cannot occur. A consideration of the second component excludes $d \leq k \leq 2d$ and $4d \leq k \leq 5d$. Thus, the only remaining option is $2d \leq k \leq 3d$ and $5d \leq l \leq 6d$, i. e.

$$\begin{pmatrix} 2d - k \\ 3d - k \end{pmatrix} = \begin{pmatrix} l - 5d \\ l - 6d \end{pmatrix},$$

implying the contradictory statements $7d = k + l = 9d$. Thus, the given path is a hexagonal polygon path. $\square$

Since the hexagonal lattice surface can be embedded into the plane, it is only natural that this embedding maps hexagonal polygon paths to `Jordan`–curves.

Remark 7.2.3. *Let $P = (p_1, p_2, \ldots, p_n)$ be a hexagonal polygon path and $\iota_H : H \to \mathbb{R}^2$ the hexagonal lattice embedding. Then the path $\rho_P : [1, n] \to \mathbb{R}^2$ with*

$$\rho_P(x) := (1 - x + \lfloor x \rfloor) \cdot \iota_H(p_{\lfloor x \rfloor}) + (x - \lfloor x \rfloor) \cdot \iota_H(p_{\lfloor x \rfloor + 1})$$

describes a continuous, closed non–intersecting curve in $\mathbb{R}^2$.

By the `Jordan`–curve–theorem, its complement has two connected components, of which exactly one is bounded.

Proof. Let $k \in \mathbb{Z}$ with $k \leq x < k + 1$, then

$$\rho_P(x) = (1 - (x - k))\iota_H(p_k) + (x - k)\iota_H(p_{k+1}).$$

Thus, ρ_P is continuous on the interval $[k, k + 1)$. To show continuity at $k + 1$, consider

$$\lim_{x \to k+1} (1 - (x - k))\iota_H(p_k) + (x - k)\iota_H(p_{k+1}) = \iota_H(p_{k+1}) = \rho_P(k + 1).$$

Thus, ρ_P is continuous.

To prove that the path is closed, it is sufficient to show that $\rho_P(1) = \iota_H(p_1)$ and $\rho_P(n) = \iota_H(p_n)$ coincide. Since $(p_1, p_2, \ldots, p_n)$ is a closed path by Definition 7.2.1, $p_1 = p_n$ holds.

Since $(p_1, p_2, \ldots, p_n)$ is non–intersecting by Definition 7.2.1 and ι_H is an embedding, $\rho_P(i) \neq \rho_P(j)$ for $1 \leq i < j < n$. Since $\rho_P([k, k + 1])$ is a subset of the edge between p_k and p_{k+1}, and the edges are disjoint, ρ_P is non–intersecting.

Proofs of the `Jordan`–curve–theorem can be found in many sources, for example [69, Theorem 1.1], [36, Proposition 2B.1], [59, Theorem 6.35], and [62, Theorem 15]. □

Remark 7.2.3 allows us to lift the `Jordan`–curve–theorem to the discrete setting of hexagonal lattice surfaces. In particular, it allows us to define the *interior faces* of a hexagonal polygon path.

Definition 7.2.4. *Let $P = (p_1, p_2, \ldots, p_n)$ be a hexagonal polygon path in the hexagonal lattice surface $H = (V, E, F, \eta, \varphi)$. Define the* ***interior faces of*** *P as the set of faces that are mapped by ι_H to the bounded connected component of $\mathbb{R}^2 \backslash \rho_P([1, n])$, with ρ_P defined as in Remark 7.2.3.*

Define the ***inner degree*** *$d_P(p_i)$ as the number of interior faces adjacent to p_i.*

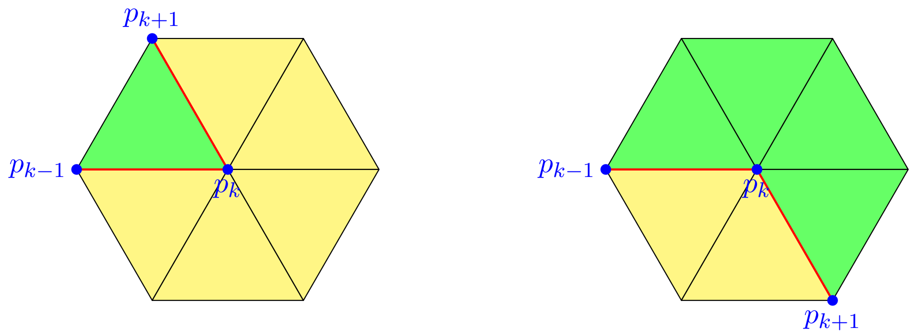
From the face partition in Definition 7.2.4, we can deduce certain combinatorial properties of inner edges. These are based on the observation that an edge lying on the hexagonal polygon path is incident to a face on the inside and one on the outside of the hexagonal polygon path.

Lemma 7.2.5. *Let P be a hexagonal polygon path. Let f be an interior face of P and g a face adjacent to f. The edge between them lies on the path P if and only if g is not an interior face.*

Proof. In the embedding, there is a continuous path between the faces. If the edge between f and g does not lie on the path P, the continuous path does not cross the closed curve from Remark 7.2.3, so f and g lie in the same connected component of the complement.

If the edge between them lies on the path, the continuous path crosses the path once, which means a switching of components. $\square$

Next, we relate the vertices in a hexagonal polygon path to the inner degrees. Let $P = (p_1, p_2, \dots, p_n)$ be a hexagonal polygon path and consider the following two situations (where green faces should be interpreted as interior faces):



In both cases, we have $p_k - p_{k-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We search for a relation between the difference $p_{k+1} - p_k$ and the inner degree $d_P(p_k)$. In the left case, we have $p_{k+1} - p_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and in the right case, $p_{k+1} - p_k = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. If we define the rotation matrix

$$R := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

we can write $p_{k+1} - p_k = R^2(p_k - p_{k-1})$ in the left case and $p_{k+1} - p_k = R^{-1}(p_k - p_{k-1})$ in the right case. This seems to generalise to the rule

$$p_{k+1} - p_k = R^{3-d_P(p_k)}(p_k - p_{k-1}).$$

However, this argument relies on the choice of interior faces. Had we interpreted the yellow faces as the interior ones, we would conclude the rule

$$p_{k+1} - p_k = R^{-3+d_P(p_k)}(p_k - p_{k-1}).$$

Fortunately, only one of these rules applies for the complete hexagonal polygon path, which we show in the next lemma.

Lemma 7.2.6. *Let $P = (p_1, p_2, \ldots, p_n)$ be a hexagonal polygon path. Let*

$$R := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then one of these two cases holds:

- $p_{i+1} - p_i = R^{3-d_P(p_i)}(p_i - p_{i-1})$ *for all i.*
- $p_{i+1} - p_i = R^{-3+d_P(p_i)}(p_i - p_{i-1})$ *for all i.*

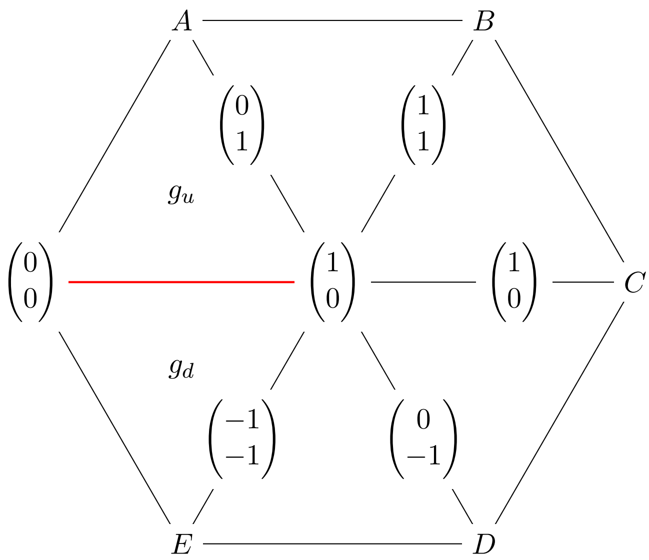
Proof. Let e_i be the edge in H that is incident to p_i and p_{i+1}. By Lemma 7.2.5:

- For each e_i, there is exactly one face f_i incident to e_i, that is an interior face of P.
- The umbrella of p_i is separated into two umbrella paths by removing e_{i-1} and e_i. The faces f_{i-1} and f_i then have to belong to the same part of the umbrella.

We know that

$$p_{i+1} - p_i \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}.$$

Since $\mathrm{Aut}(H)$ acts regularly on triples of adjacent vertices by Lemma 7.1.4 and Remark 7.1.3, it also acts transitively on pairs of adjacent vertices. Thus, without loss of generality, we set $p_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.



We differentiate according to f_1:

1. If $f_1 = g_u$, we further differentiate between the possible positions of p_3:

a) If $p_3 = A$, then $d_P(p_2) = 1$ and $p_3 - p_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b) If $p_3 = B$, then $d_P(p_2) = 2$ and $p_3 - p_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

c) If $p_3 = C$, then $d_P(p_2) = 3$ and $p_3 - p_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

d) If $p_3 = D$, then $d_P(p_2) = 4$ and $p_3 - p_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

e) If $p_3 = E$, then $d_P(p_2) = 5$ and $p_3 - p_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

Thus, $p_3 - p_2 = R^{3-d_P(p_2)}(p_2 - p_1)$.

To extend this partial result to all $1 \leq i \leq n$, we perform an automorphism of H that transforms p_2 into $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and p_3 into $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, as well as f_2 into f_1. Therefore, the same result applies to all i.

2. If $f_1 = g_d$, all degrees are replaced by $6 - \deg_P(p_2)$ in contrast to the first case. This gives the result in the second case. □

Lemma 7.2.6 is crucial to establish the main result of this section: an existence statement for hexagonal polygon paths with specified inner degrees.

Lemma 7.2.7. *Let $T = (t_1, t_2, \ldots, t_k) \in \mathbb{N}^k$ such that*

1. $1 \leq t_i \leq 5$ for all $1 \leq i \leq k$ and

2. for all $1 \leq a < b \leq k$ we have $|\sum_{i=a}^{b}(3 - t_i)| \leq 2$.

Then there exists a hexagonal polygon path $P = (p_1, p_2, \ldots, p_n)$ with $n > k$ such that $d_P(p_i) = t_i$ for all $1 \leq i \leq k$.

Proof. We construct the hexagonal polygon path explicitly. Define

$$p_1 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad p_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad p_{i+1} := p_i + R^{3-t_i}(p_i - p_{i-1})$$

for all $2 < i < k$. This defines a sequence $(p_1, p_2, \ldots, p_{k+1})$. If we can show that we extend this to a hexagonal polygon path $P = (p_1, p_2, \ldots, p_{k+1}, \ldots, p_n)$, such that the face $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is an interior face of the path, the claim follows from Lemma 7.2.6 (since $t_i = d_P(p_i)$).

Thus, we need to show that $(p_1, p_2, \ldots, p_{k+1})$ is a non–intersecting path and that it can be extended to a hexagonal polygon path. To show that the path is non–intersecting, we consider the difference vectors $d_i := p_{i+1} - p_i$ for $1 \leq i < k$. If $p_a = p_b$ for any $1 \leq a < b \leq k+1$, we would have

$$p_1 + \sum_{i=1}^{a-1} d_i = p_1 + \sum_{i=1}^{b-1} d_i.$$

This can be rewritten as $p_a = p_a + \sum_{i=a}^{b-1} d_i$. It is therefore sufficient to show that any such sum cannot be 0. By applying an appropriate power of R, we can always assume $d_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

By induction, we obtain for all $1 \leq a < b < k$

$$d_b = R^{\sum_{i=a}^{b-1}(3-t_i)} d_a. \tag{7.1}$$

By our additional assumption, this implies $d_b = R^x d_a$ with $x \in \{-2, -1, 0, 1, 2\}$. The possible vectors are

$$R^{-2} d_a = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad R^{-1} d_a = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad R^{0} d_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad R^{1} d_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad R^{2} d_a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume $\sum_{i=a}^{b-1} d_i = 0$. Since $d_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, there has to be a $a < j < b$ with $d_j = R^{-2} d_a$ (consider first component). If any d_i with $a < i < j$ fulfilled $d_i = R d_a$ or $d_i = R^2 d_a$, Equation 7.1 would make this impossible, since this implies $d_j = R^x d_i$ with $x \in \{3, -4\}$, in contradiction to our assumption The same argument applies for $j < i < b$.

But then, we have $\{d_a, d_{a+1}, \ldots, d_{b-1}\} \subseteq \{d_a, R d_a, R^2 d_a\}$. Considering the second component, all of them have to be equal to d_a. This is a contradiction to $d_j \neq d_a$. Therefore, this hexagonal path is non–intersecting.

To complete the proof, we have to extend $(p_1, \ldots, p_{k+1})$ to a hexagonal polygon path. We can extend the hexagonal path in both directions, always with inner degree 3 (this extended path also fulfils the conditions of this lemma, therefore we do not produce any new intersections). Since the path so far was bounded, it has some maximal distance D to the origin. We extend the path in both directions up to distance $D + 1$. Thus, the path has two vertices in common with the hexagonal polygon path from Example 7.2.2. We can now complete our path in two different ways, and we choose in such a way that the face $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ becomes an interior face of the resulting hexagonal polygon path. $\square$

7.3 Transition maps

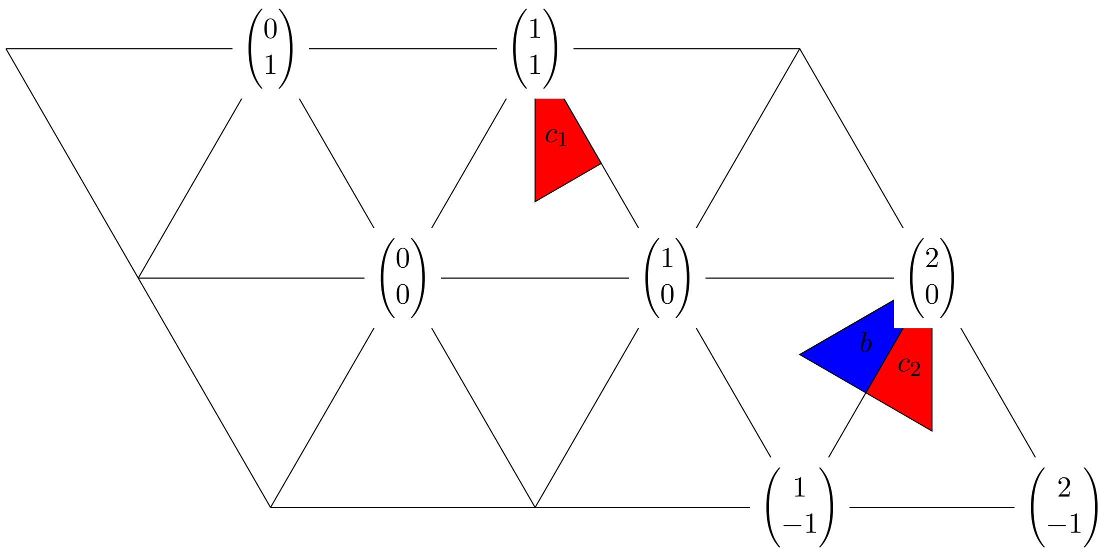
In the previous sections, we defined the hexagonal lattice surface and worked with combinatorial closed paths. In this section, we apply a differential–geometric approach: We represent a triangular combinatorial surface as a subset of the hexagonal lattice, together with certain transition maps (made precise in Definition 7.3.5).

This approach has been very successful for translation surfaces (called *origamis*), where it connects the combinatorial structure of a discrete surface build from squares with its differential–geometric properties. For an introduction, compare [70].

Inspired by this success, we analyse triangular combinatorial surfaces under this perspective as well. Conceptually, we map each face of the combinatorial surface to a face of the hexagonal lattice surface. If two faces are adjacent, there is an automorphism of the hexagonal lattice mapping the first image next to the second image. These special automorphisms are the *transition maps*.

Before we formally define transition maps, we illustrate it on an example.

Example 7.3.1. *Consider the following situation in the hexagonal lattice surface:*



We want to map the flag c_1 *to the flag* b *(2–adjacent to the flag* c_2*). For that, we need to find an* $(M,t) \in D_{12} \ltimes \mathbb{Z}^2$ *with*

$$M\begin{pmatrix}1\\1\end{pmatrix}+t=\begin{pmatrix}2\\0\end{pmatrix} \qquad M\begin{pmatrix}1\\0\end{pmatrix}+t=\begin{pmatrix}1\\-1\end{pmatrix} \qquad M\begin{pmatrix}0\\0\end{pmatrix}+t=\begin{pmatrix}1\\0\end{pmatrix}.$$

This is satisfied by

$$M = r^{-1} = \begin{pmatrix}0 & 1\\-1 & 1\end{pmatrix} \qquad t=\begin{pmatrix}1\\0\end{pmatrix}.$$

We generalise the process from Example 7.3.1 into the definition of *transition map*.

Definition 7.3.2. *Let* H *be the hexagonal lattice surface and* (v_1,e_1,f_1) *and* (v_2,e_2,f_2) *be flags. The unique* $\alpha \in \mathrm{Aut}(S)$ *with*

$$\alpha(v_1)=v_2 \qquad \alpha(e_1)=e_2 \qquad \alpha(f_1)\neq f_2$$

is called ***transition map*** *from* (v_1,e_1,f_1) *to* (v_2,e_2,f_2).

Well–defined. Since H is a closed surface, there is exactly one flag (v_2, e_2, g) with $f_2 \neq g$ by Remark 2.7.11. Since $D_{12} \ltimes \mathbb{Z}^2$ acts regular on the flags, there is exactly one such map α (compare Lemma 7.1.4 and Remark 7.1.3). □

If we know the transition map from a flag c_1 to a flag c_2, we can also describe the transition map from c_2 to c_1.

Remark 7.3.3. *Let H be the hexagonal lattice surface and α the transition map from the flag c_1 to the flag c_2. Then, α^{-1} is the transition map from the flag c_2 to the flag c_1.*

Although we defined transition maps on flags, their primary importance comes from mapping faces.

Remark 7.3.4. *Let H be the hexagonal lattice surface and α the transition map from the flag c_1 to the flag c_2. Then, the face $\alpha(\lambda_2(c_1))$ is adjacent to the face $\lambda_2(c_2)$, via the edge $\lambda_1(c_2)$.*

At this point, we can define the differential–geometric structure of a twisted triangular surface. Intuitively, this corresponds to the following construction:

1. Map every face of the twisted triangular surface S to a face of the hexagonal lattice surface H. Formally, we represent this as a map from the chambers of S to the flags of H, such that faces are preserved.
2. Then, we need to preserve the adjacencies between faces. If two adjacent faces in S are mapped to some non–adjacent faces, we need a transition map to "shift" one of them so that it becomes adjacent to the other one.

In total, this describes the twisted triangular surface S.

Definition 7.3.5. *Let $S = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be a twisted triangular surface and H the hexagonal lattice surface with flags $\mathcal{F}$ and involutions σ_0^H and σ_1^H from Remark 2.5.11 and Remark 2.5.12*

A ***net*** *of S is a map $\rho : C \to \mathcal{F}$ satisfying*

$$\rho(\sigma_0(c)) = \sigma_0^H(\rho(c)) \qquad\qquad \rho(\sigma_1(c)) = \sigma_1^H(\rho(c))$$

for all chambers $c \in C$.

For each $c \in C$, let α_c be the transition map from $\rho(c)$ to $\rho(c^*)$ with $[c]_\sim = \{c, c^*\}$. The ***transition group*** *of the net is the subgroup of $D_{12} \ltimes \mathbb{Z}^2$ generated by $\{\alpha_c \mid c \in C\}$*

We emphasize that there are several nets for each twisted triangular surface, which can differ substantially.

Example 7.3.6. *Consider the twisted triangular surface $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ with*

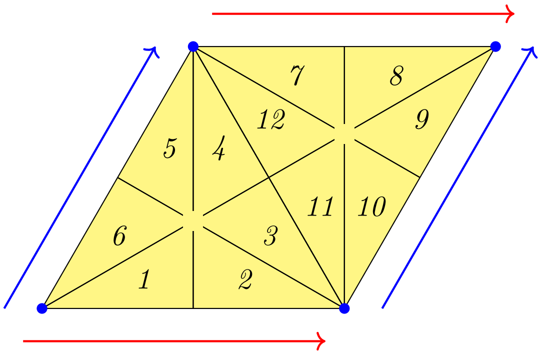
$$V = \{v\}, \qquad E = \{e_1, e_2, e_3\}, \qquad F = \{f_1, f_2\}, \qquad C = \{c_1, \dots, c_{12}\},$$

and

$$\lambda : C \to V \times E \times F, \qquad c_k \mapsto \begin{cases} (v, e_1, f_1) & k \in \{1,2\} \\ (v, e_2, f_1) & k \in \{3,4\} \\ (v, e_3, f_1) & k \in \{5,6\} \\ (v, e_1, f_2) & k \in \{7,8\} \\ (v, e_3, f_2) & k \in \{9,10\} \\ (v, e_2, f_2) & k \in \{11,12\} \end{cases}$$

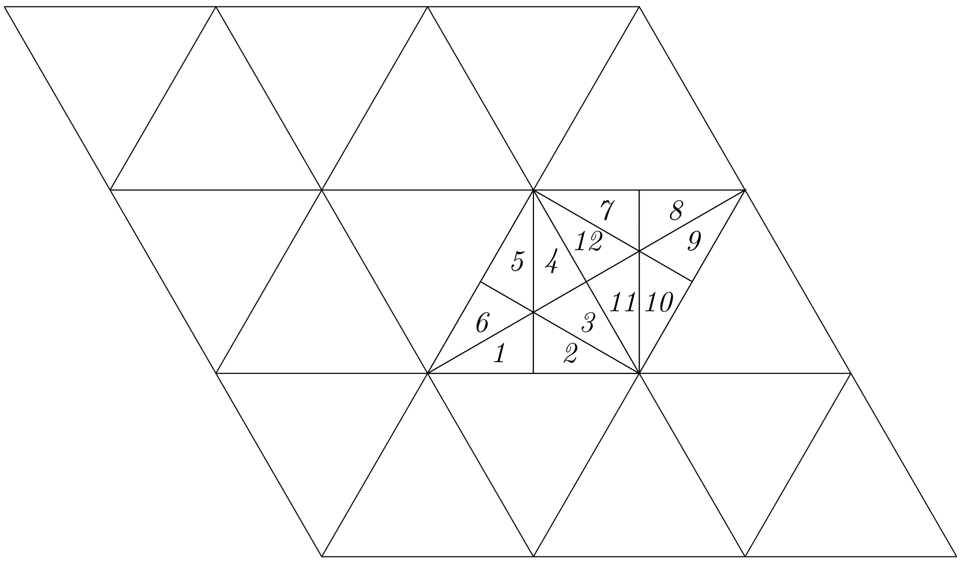
$$\sigma_0 = (c_1, c_2)(c_3, c_4)(c_5, c_6)(c_7, c_8)(c_9, c_{10})(c_{11}, c_{12})$$
$$\sigma_1 = (c_1, c_6)(c_2, c_3)(c_4, c_5)(c_7, c_{12})(c_8, c_9)(c_{10}, c_{11})$$
$$\sim : \{c_1, c_7\}, \{c_2, c_8\}, \{c_3, c_{11}\}, \{c_4, c_{12}\}, \{c_5, c_9\}, \{c_6, c_{10}\},$$

that is illustrated by



We now consider two different nets.

1. *Consider the net* $\rho : C \to C_H$*, like in this illustration:*

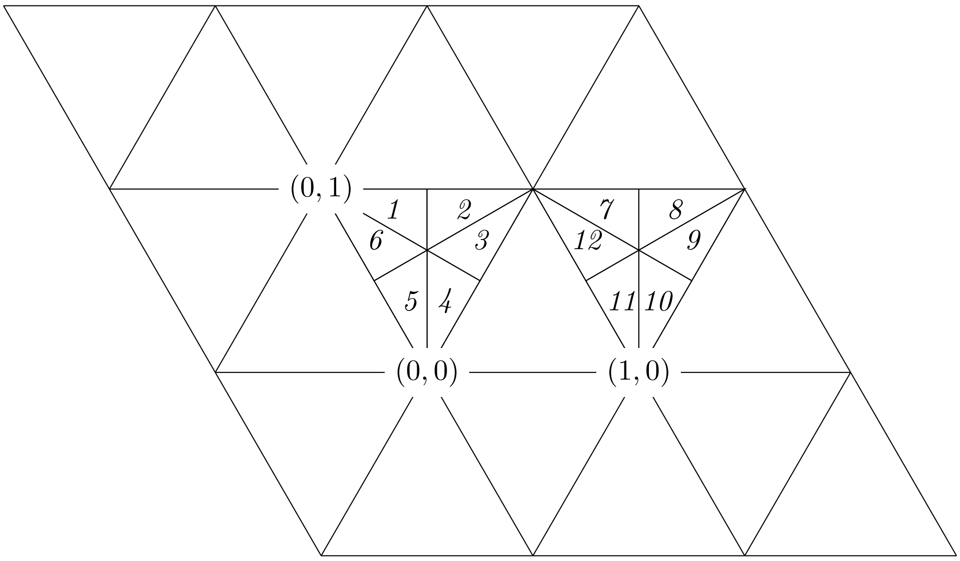


The transition maps are translations:

$$+\begin{pmatrix}0\\0\end{pmatrix} \qquad \pm\begin{pmatrix}1\\0\end{pmatrix} \qquad \pm\begin{pmatrix}0\\1\end{pmatrix}$$

Thus, the transition group of this net is $\mathbb{Z}^2$.

2. *Consider the net* $\rho : C \to C_H$*, like in this illustration:*



We compute the transition maps (M, t) *between the flags:*

$$1 \mapsto 7 \qquad M = rm \qquad t = \begin{pmatrix}2\\2\end{pmatrix}$$

$$3 \mapsto 11 \qquad M = rm \qquad t = \begin{pmatrix}1\\1\end{pmatrix}$$

$$5 \mapsto 9 \qquad M = rm \qquad t = \begin{pmatrix}2\\1\end{pmatrix}$$

Example 7.3.6 illustrates that the transition group heavily depends on the concrete net ρ. In particular, if the twisted triangular surface has sufficiently many faces, we can construct a net that has $D_{12} \ltimes \mathbb{Z}^2$ as its transition group.

But like with origamis (mentioned at the start of Section 7.3), it is not important that there are arbitrarily horrible nets. Rather, it is important that we can find a net that is *nice*. In the remainder of this section, we characterise which surfaces have "nice" nets.

Obviously, there are several different ways in which a transition group can be nice. For origamis, the transition group only consists of translations. Since it does not matter if it is the full translation subgroup $\mathbb{Z}^2$, we generalise in the following way: We consider the projection onto the first component of $D_{12} \ltimes \mathbb{Z}^2$. In Example 7.3.6, this projection is trivial for the first net and $\langle rm \rangle$ for the second one.

Remark 7.3.7. *Let H be the hexagonal lattice surface and ρ a net with transition group T. For any automorphism $\alpha \in \operatorname{Aut}(H)$, the transition group of $\alpha \circ \rho$ is $\alpha T \alpha^{-1}$.*

Remark 7.3.7 shows that the projection onto the first component only has to be analysed up to conjugation. For simplicity, we focus our attention on normal subgroups.

Remark 7.3.8. *The normal subgroup lattice of D_{12} has the following form:*

$$\begin{tikzcd}
 & D_{12} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle r \rangle \cong C_6 \arrow[dr, dash] \arrow[dd, dash] & \langle r^2, m \rangle \cong S_3 \arrow[d, dash] & \langle r^2, mr \rangle \cong S_3 \arrow[dl, dash] \\
 & \langle r^2 \rangle \cong C_3 \arrow[dd, dash] & \\
\langle r^3 \rangle \cong C_2 \arrow[dr, dash] & & \\
 & 1 &
\end{tikzcd}$$

We are interested in relating possible transition groups with combinatorial properties of the surface. This is only possible if we find properties that interact nicely with the automorphism group. We consider the following properties:

- *Face–2–colouring* from Definition 3.3.2. It is a map $F \to \{1, 2\}$.
- *Orientability* from Definition 5.3.5 and *dual orientability* from Definition 5.3.6. Both are maps $C \to \{\pm 1\}$.
- *RRR–colouring* from Definition 3.3.3. It is a map $E \to \{1, 2, 3\}$.

All of these structures are uniquely determined if they are defined on one face. Thus, if we find an automorphism that preserves this structure on one face, it preserves the structure on all faces. We start by understanding which automorphisms preserve which structure.

Remark 7.3.9. *Let H be the hexagonal lattice surface. It has a face–2–colouring via $\{p, p^{+0}, p^{++}\} \mapsto 1$ and $\{p, p^{++}, p^{0+}\} \mapsto 2$. An $\alpha \in \operatorname{Aut}(H)$ preserves this face–2–colouring if and only if $\alpha \in \langle r^2, m \rangle \ltimes \mathbb{Z}^2$.*

Proof. $\alpha = (M, t) \in D_{12} \ltimes \mathbb{Z}^2$ preserves the face–2–colouring if and only if α maps the face $\{p, p^{+0}, p^{++}\}$ to a face of the form $\{q, q^{+0}, q^{++}\}$.

By Remark 7.1.5, $M = r^k m$ or $M = r^k$ with $0 \leq k \leq 5$. From the proof of Lemma 7.1.4, we obtain that the action of r swaps the two face types, while the action of m leaves them invariant. Thus, k has to be even. □

Remark 7.3.10. *Let H be the hexagonal lattice surface. It is orientable and $\alpha \in \mathrm{Aut}(H)$ preserves the orientability if and only if $\alpha \in \langle r \rangle \ltimes \mathbb{Z}^2$.*

Proof. We describe the orientation of H by 3–cycles of vertices (a local orientation map), according to Definition 5.3.2:

$$\{p, p^{+0}, p^{++}\} \mapsto (p, p^{+0}, p^{++}) \qquad\qquad \{p, p^{++}, p^{0+}\} \mapsto (p, p^{++}, p^{0+})$$

$\alpha = (M, t) \in D_{12} \ltimes \mathbb{Z}^2$ preserves the orientability if and only if α preserves the local orientation map. We check the generators r and m. From the proof of Lemma 7.1.4, we obtain

$$\begin{aligned} r.(p, p^{+0}, p^{++}) &= (rp, (rp)^{++}, (rp)^{0+}) & m.(p, p^{+0}, p^{++}) &= (mp, (mp)^{0-}, (mp)^{--}) \\ r.(p, p^{++}, p^{0+}) &= (rp, (rp)^{0+}, (rp)^{-0}) & m.(p, p^{++}, p^{0+}) &= (mp, (mp)^{--}, (mp)^{-0}). \end{aligned}$$

The action of r preserves the orientation, the action of m inverses it. Thus, $M \in \langle r, m^2 \rangle = \langle r \rangle$. □

Remark 7.3.11. *Let H be the hexagonal lattice surface. It is dual orientable and $\alpha \in \mathrm{Aut}(H)$ preserves the dual orientability if and only if $\alpha \in \langle r^2, mr \rangle \ltimes \mathbb{Z}^2$.*

Proof. We describe the dual orientation of H by 3–cycles of vertices (a local orientation map), according to Definition 5.3.3:

$$\{p, p^{+0}, p^{++}\} \mapsto (p, p^{+0}, p^{++}) \qquad\qquad \{p, p^{++}, p^{0+}\} \mapsto (p, p^{0+}, p^{++})$$

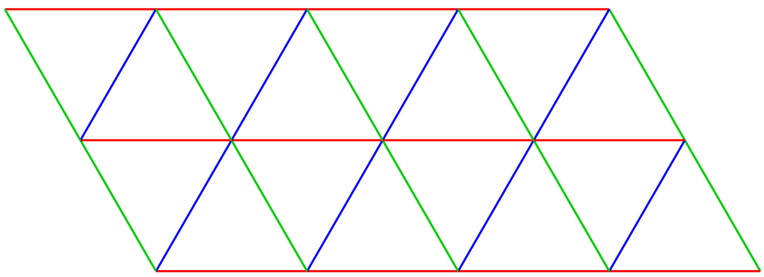
$\alpha = (M, t) \in D_{12} \ltimes \mathbb{Z}^2$ preserves the orientability if and only if α preserves the local orientation map. By Remark 7.1.5, $M = r^k$ or $M = r^k m$ with $0 \leq k \leq 5$. We check the generators r and m. From the proof of Lemma 7.1.4, we obtain

$$\begin{aligned} r.(p, p^{+0}, p^{++}) &= (rp, (rp)^{++}, (rp)^{0+}) & m.(p, p^{+0}, p^{++}) &= (mp, (mp)^{0-}, (mp)^{--}) \\ r.(p, p^{0+}, p^{++}) &= (rp, (rp)^{-0}, (rp)^{0+}) & m.(p, p^{0+}, p^{++}) &= (mp, (mp)^{-0}, (mp)^{--}). \end{aligned}$$

Both r and m invert the dual orientation. Thus, we have

$$M \in \{r^0, r^2, r^4, rm, r^3 m, r^5 m\} = \langle r^2, rm \rangle.$$ □

The hexagonal lattice surface has an RRR–colouring, illustrated here:



Remark 7.3.12. *Let $H = (V, E, F, \eta, \varphi)$ be the hexagonal lattice surface from Definition 7.1.2. The map $c : E \to \{1, 2, 3\}$ with*

$$c(\{p, p^{+0}\}) := 1 \qquad c(\{p, p^{++}\}) := 2 \qquad c(\{p, p^{0+}\}) := 3$$

is an RRR–colouring of H. An automorphism $\alpha \in \mathrm{Aut}(H)$ preserves this colouring if and only if $\alpha \in \langle r^3\rangle \ltimes \mathbb{Z}^2$.

Proof. Since each edge of H falls into exactly one of the three edge types, the map c is well–defined. Clearly, the three edges of a face have different types, so c is a Grünbaum colouring.

It remains to show that all local symmetries are of type R. Since Definition 3.3.4 is formulated for twisted polygonal complexes, we need to transfer H into a twisted polygonal complex $(V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ with the functor $\mathsf{TwistPoly}$ from Definition 2.5.13.

Consider the edge $\{p, p^{+0}\}$ with $p \in V$. Two $\sim$–equivalent chambers containing this edge are

$$(p, \{p, p^{+0}\}, \{p, p^{+0}, p^{++}\}) \qquad (p, \{p, p^{+0}\}, \{p, p^{+0}, p^{0-}\}).$$

Applying σ_1 leads to the chambers

$$(p, \{p, p^{++}\}, \{p, p^{+0}, p^{++}\}) \qquad (p, \{p, p^{0-}\}, \{p, p^{+0}, p^{0-}\}).$$

Clearly, the edges $\{p, p^{++}\}$ and $\{p, p^{0-}\}$ have different images under c_E.

The arguments for the other edge types are analogous.

It remains to show that $\alpha = (M, t) \in D_{12} \ltimes \mathbb{Z}^2$ preserves this colouring if and only if $M \in \langle r^3\rangle$. By Remark 7.1.5, $M = r^k m$ or $M = r^k$ with $0 \leq k \leq 5$. We consider the proof of Lemma 7.1.4 to understand the action of r and m on the edges. The element r induces a 3–cycle on the edge types, the element m exchanges two types while leaving the third one fixed. Thus, the case $M = r^k$ is possible if and only if $k \in \{0, 3\}$. The case $M = r^k m$ is not possible, since the single type fixed by m also has to be fixed by r^k. But this is only possible if r^k fixes all types. ☐

For convenience, we give these properties a name.

Definition 7.3.13. *We call face–2–colourability, orientability, dual orientability, and RRR–colourability* ***heritable properties****.*

Heritable properties are special since they can be "inherited" from the hexagonal lattice surface to twisted triangular surfaces with specific nets.

Theorem 7.3.14. *Let S be a twisted triangular surface and H the hexagonal lattice surface.*

1. *If S has a net whose transition group preserves a heritable property of H, then S also has this property.*

2. *If S has a heritable property, there exists a net of S whose transition group preserves this property of H.*

Proof. Start with the first claim: We define the corresponding structure on (the faces of) S such that it is compatible with the structure of the net (in H). Then, we have to check whether this definition is compatible along adjacent faces of S. Pick two adjacent faces in S and let f and g be the corresponding faces in H. There is an element of the transition group that maps f to an adjacent face of g (Remark 7.3.4). By assumption, the structure of f is preserved under this map. Thus, the compatibility follows from the compatibility relation in H.

Conversely, assume S has a heritable structure. Since H also has this structure, we define the map ρ of a net of S (compare Definition 7.3.5) in such a way that the structure is preserved. Then, all transition maps have to map the faces of H in such a way as to preserve the structure. This implies already that each such transition map preserves the full heritable property of H. $\square$

With Theorem 7.3.14 in hand, we can associate a property to each normal subgroup of D_{12}. The result is depicted in Figure 7.1. For convenience, we define the special case

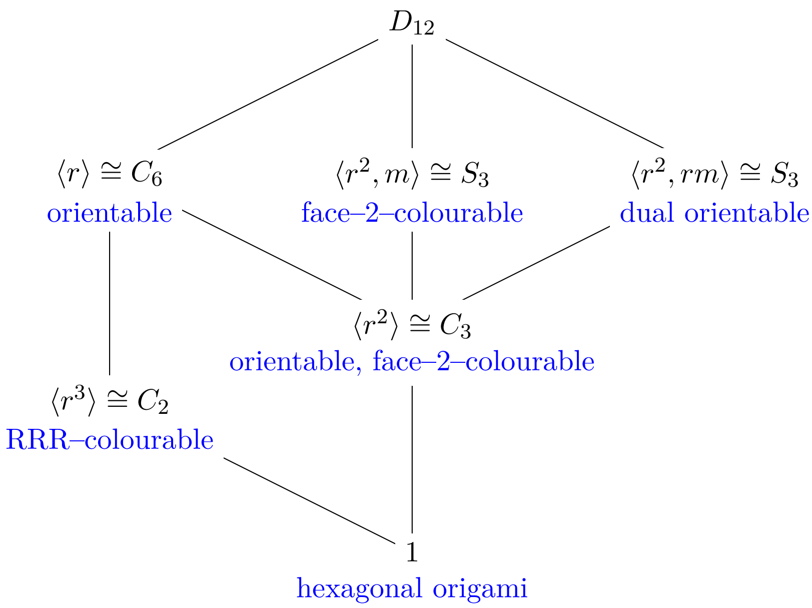


Figure 7.1: The normal subgroup lattice of D_{12} as a property lattice.

from the start of Section 7.3.

Definition 7.3.15. *A twisted triangular surface is called* ***hexagonal origami*** *if it has a net whose transition group only consists of translations.*

Figure 7.1 depicts a *property lattice*, a lattice whose elements are different properties. We recall the definition of a lattice (longer treatments can be found in [20] and [26]):

Definition 7.3.16. *A* ***lattice*** $(M, \leq)$ *consists of a set* M *and a binary relation* $\leq$ *on* M*, such that:*

1. ***reflexivity****:* $m \leq m$ *for all* $m \in M$*.*
2. ***transitivity****:* $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$ *for all* $x, y, z \in M$*.*
3. ***antisymmetry****:* $x \leq y$ *and* $y \leq x$ *imply* $x = y$ *for all* $x, y \in M$*.*
4. ***joins exist****: For* $x, y \in M$*, there is a unique element in* M*, called* $x \vee y$ *(or* ***join****) such that*
   - $x \leq x \vee y$ *and* $y \leq x \vee y$*.*
   - *Any* $z \in M$ *with* $x \leq z$ *and* $y \leq z$ *satisfies* $x \vee y \leq z$*.*
5. ***meets exist****: For* $x, y \in M$*, there is a unique element in* M*, called* $x \wedge y$ *(or* ***meet****) such that*
   - $x \wedge y \leq x$ *and* $x \wedge y \leq y$*.*
   - *Any* $z \in M$ *with* $z \leq x$ *and* $z \leq y$ *satisfies* $z \leq x \wedge y$*.*

A lattice becomes a property lattice if we can associate a property to each of its elements such that the logical relations of the properties correspond to the lattice structure.

Definition 7.3.17. *A* ***property lattice*** *is a lattice* $(M, \leq)$*, where each* $m \in M$ *corresponds to a boolean property of a twisted triangular surface, such that for every twisted polygonal surface* S *holds:*

1. *If* S *has the property of an* $m \in M$*, it also has the property of* $n \in M$ *if* $m \leq n$*.*
2. *If* S *has the properties of* $m \in M$ *and* $n \in M$*, it also has the properties of their meet* $m \wedge n$*.*

Now, we can formally prove that Figure 7.1 depicts a property lattice.

Lemma 7.3.18. *The normal subgroup lattice of* D_{12}*, together with the properties shown in Figure 7.1, is a property lattice.*

Proof. For a twisted triangular surface S, Theorem 7.3.14 connects properties of S and whether there exists a net whose transition group lies in a certain subgroup of $\mathrm{Aut}(H)$.

In Remark 7.3.10, Remark 7.3.9, Remark 7.3.11, and Remark 7.3.12, we show specifically which properties correspond to which normal subgroups.

Since all of these were formulated as "$\alpha \in U$", for a subgroup $U \leq \mathrm{Aut}(H)$, the first requirement of Definition 7.3.17 is satisfied.

For the second one, assume that S has two properties. Then, we can apply Theorem 7.3.14 to the *combination* of the corresponding structures. This gives a group intersection in $\mathrm{Aut}(H)$ since we have to compute which automorphisms preserve both structures in H. □

7.3.1 Extended property lattice

In Section 7.3, we used nets with certain properties to relate different properties of twisted triangular surfaces with each other. This culminated in the construction of the property lattice of Figure 7.1.

In this subsection, we extend this property lattice by adding MMM–colourability from Definition 3.3.3. The extended property lattice is depicted in Figure 7.2.

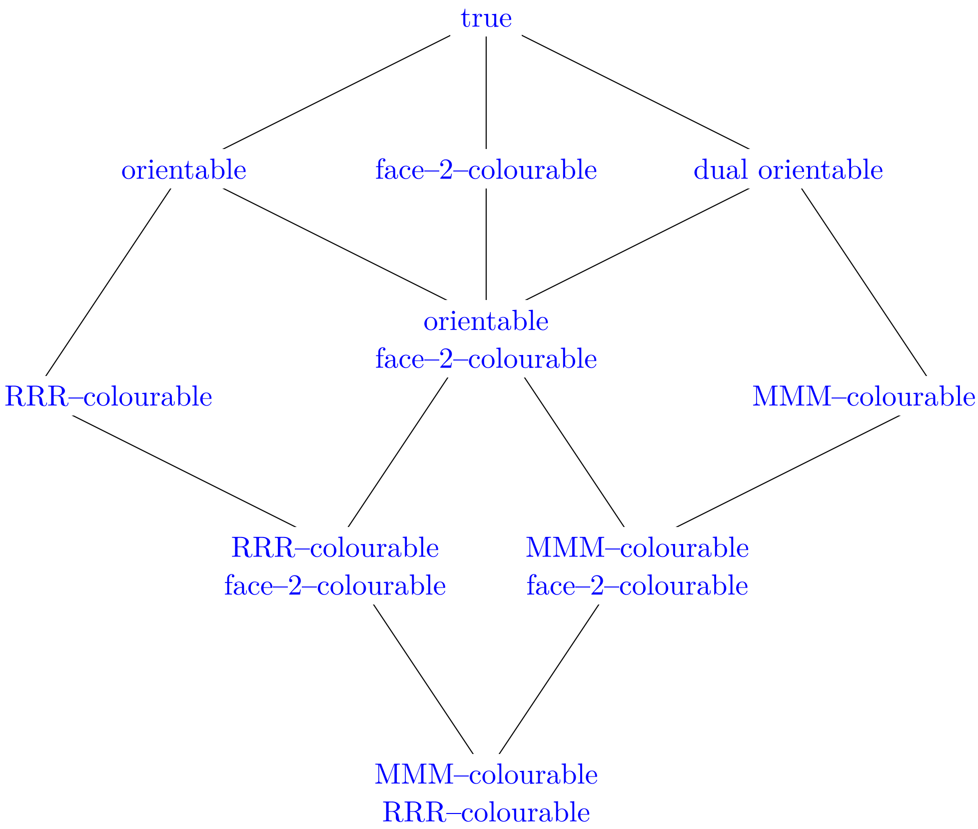


Figure 7.2: Extended property lattice

To prove that Figure 7.2 depicts a property lattice, we need to show the connection between MMM–colourings and dual orientations.

Lemma 7.3.19. *An MMM–colourable twisted triangular surface is dual orientable.*

Proof. Let $S = (V, E, F, C, \lambda, \sigma_0, \sigma_1, \sim)$ be the twisted triangular surface and $c_E : E \to \{1, 2, 3\}$ the MMM–colouring. Define a map

$$s : C \to \{\pm 1\} \qquad c \mapsto \begin{cases} +1 & c_E(\lambda_1(c)) + 1 = c_E(\lambda_1(\sigma_1(c))) \mod 3 \\ -1 & c_E(\lambda_1(c)) - 1 = c_E(\lambda_1(\sigma_1(c))) \mod 3 \end{cases}$$

We show that s is a local chamber colouring (Definition 5.3.4). Clearly, $s(c) \neq s(\sigma_1(c))$ for all chambers $c \in C$. Since S has a Grünbaum colouring, $\lambda_1(\sigma_1(c)) \neq \lambda_1(\sigma_1\sigma_0(c))$. Since $\lambda_1(c) = \lambda_1(\sigma_0(c))$, this implies $s(c) \neq s(\sigma_0(c))$.

It remains to show that 2–adjacent chambers have equal s–value. Assume $c_1 \sim c_2$ are different chambers. Thus, $\lambda_1(c_1) = \lambda_1(c_2)$. Since c_E is an MMM–colouring, Definition 3.3.4 gives $c_E(\lambda_1(\sigma_1(c_1))) = c_E(\lambda_1(\sigma_1(c_2)))$, which proves the claim. ☐

Now, we can prove that Figure 7.2 depicts a property lattice.

Lemma 7.3.20. *The property lattice depicted in Figure 7.2 is a property lattice.*

Proof. We check the conditions of Definition 7.3.17. Let S be a twisted triangular surface. We start with the first condition:

- Assume S has an MMM–colouring. We have to show that S is dual orientable. This follows from Lemma 7.3.19.

- If S has an MMM–colouring and an RRR–colouring, it is also dual orientable (Lemma 7.3.19). Thus, by Lemma 7.3.18, it is also a hexagonal origami and face–2–colourable.

Consider the second condition now. In comparison to the property lattice of Figure 7.1, we only have to consider MMM–colourable surfaces with an additional property. The only non–trivial newly available meet is a twisted triangular surface that is both MMM–colourable and orientable. By Lemma 7.3.19, it is also dual orientable. By Lemma 7.3.18, S is face–2–colourable. ☐

The lattice in Figure 7.2 has a mirror symmetry. This is not a coincidence. In Chapter 9, we introduce the notion of geodesic duality. One can show the following statements for a twisted triangular surface S:

1. S is orientable if and only if its dual is dual orientable.

2. S is RRR–colourable if and only if its dual is MMM–colourable.

3. S is face–2–colourable if and only if its dual is face–2–colourable.

Thus, the mirror symmetry apparent in Figure 7.2 reflects a duality within twisted triangular surfaces.

8 Invariants of simplicial surfaces with single boundary

In the analysis of simplicial surfaces, a common obstacle is the sheer variance of surfaces. Many of them look very similar, if one chooses to eschew a few details, yet show very different behaviour. This apparent lack of structure hinders many naive approaches to understand and classify them properly.

The goal of this chapter is the description of a global structure within simplicial surfaces, i. e. a structure that remains invariant under local modifications of the surface. This structure should give a better perspective on the nature of simplicial surfaces.

In Section 8.1, we analyse a particular local modification that serves as motivation for developing the general theory of global invariants. To define these global invariants, we define the *infinite regular extension* in Section 8.2. In Section 8.3, we construct several infinite regular extensions for different surfaces and analyse the different invariants that are induced by them. These invariants rely on the classification of possible infinite regular extensions, which mirrors the classification of nanotubes and nanocones (compare [17] and [18] for more details).

8.1 Motivation: Vertex splitting

It is known from [57] that all spherical triangulations can be obtained from the tetrahedron by successively inserting two triangles (called *vertex splits* in the literature, compare [35, D22, Subsection 7.8.3]). There is a rich literature surrounding this topic. For example, [9] proves a similar result for triangulations of the projective plane.

Vertex splits can be described as follows:

1. Choose two edges that are incident to the same vertex.
2. Cut along these edges. This leaves a hole with four boundary edges.
3. Insert two triangles into the hole.

We can illustrate a vertex split as in Figure 8.1. It is not immediately clear how to apply these steps on vertex–faithful simplicial surfaces. In particular, the cutting operation is unclear.

Definition 8.1.1. *Let $S = (V, E, F, \eta, \varphi)$ be a simplicial surface and e_1, e_{k+1} two edges with $\eta(e_1) = \{v_1, v_2\}$ and $\eta(e_{k+1}) = \{v_2, v_3\}$, such that v_2 is an inner vertex with umbrella–path $(e, f_1, e_2, \ldots, e_k, f_k, e_{k+1}, f_{k+1}, e_{k+2}, \ldots, e_n, f_n, e)$.*

The ***vertex split of*** S ***along*** $\{e_1, e_{k+1}\}$ *is the simplicial surface $(\bar{V}, \bar{E}, \bar{F}, \bar{\eta}, \bar{\varphi})$ with*

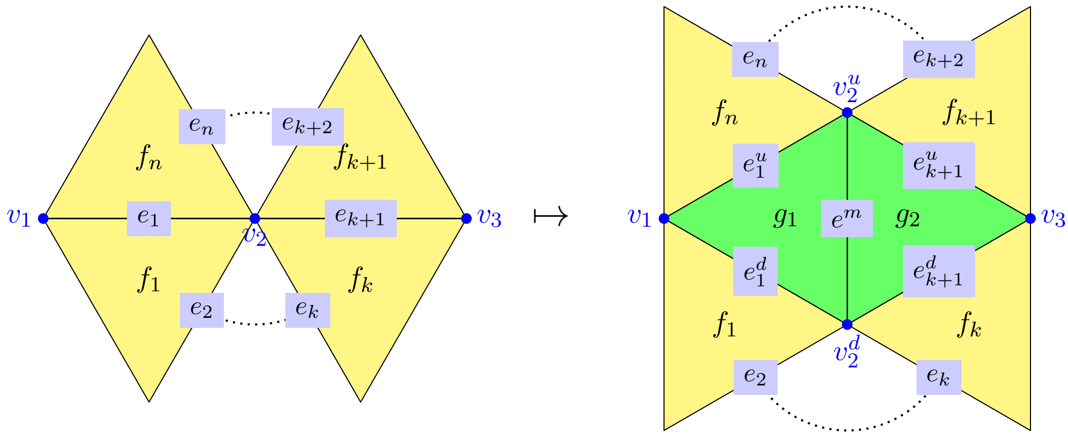


Figure 8.1: Vertex split

- $\bar{V} := V \backslash \{v_2\} \cup \{v_2^u, v_2^d\}$,
- $\bar{E} := E \backslash \{e_1, e_{k+1}\} \cup \{e_1^d, e_1^u, e_{k+1}^d, e_{k+1}^u, e^m\}$,
- $\bar{F} := F \cup \{g_1, g_2\}$,
- $$\bar{\eta} : \bar{E} \to \mathrm{Pot}_2(\bar{V}), x \mapsto \begin{cases} \{v_2^u, v_2^d\} & x = e^m \\ \{v_1, v_2^u\} & x = e_1^u \\ \{v_1, v_2^d\} & x = e_1^d \\ \{v_3, v_2^u\} & x = e_{k+1}^u \\ \{v_3, v_2^d\} & x = e_{k+1}^d \\ \eta(x) \backslash \{v_2\} \cup \{v_2^u\} & x \in \{e_{k+2}, \dots, e_n\} \\ \eta(x) \backslash \{v_2\} \cup \{v_2^d\} & x \in \{e_2, \dots, e_k\} \\ \eta(x) & \textit{otherwise,} \end{cases}$$
- *and*
$$\bar{\varphi} : \bar{F} \to \mathrm{Pot}_3(\bar{E}), x \mapsto \begin{cases} \{e_1^u, e_1^d, e^m\} & x = g_1 \\ \{e_{k+1}^u, e_{k+1}^d, e^m, \} & x = g_2 \\ \varphi(x) \backslash \{e_1\} \cup \{e_1^u\} & x = f_n \\ \varphi(x) \backslash \{e_1\} \cup \{e_1^d\} & x = f_1 \\ \varphi(x) \backslash \{e_{k+1}\} \cup \{e_{k+1}^u\} & x = f_{k+1} \\ \varphi(x) \backslash \{e_{k+1}\} \cup \{e_{k+1}^d\} & x = f_k \\ \varphi(x) & \textit{otherwise.} \end{cases}$$

Well–defined. We show first that $(\bar{V}, \bar{E}, \bar{F}, \bar{\eta}, \bar{\varphi})$ is a triangular complex, using Definition 2.5.2.

1. For the face g_1, we have the sequence $(v_1, e_1^u, v_2^u, e^m, v_2^d, e_1^d)$. For the face g_2, we have the sequence $(v_3, e_{k+1}^u, v_2^u, e^m, v_2^d, e_{k+1}^d)$.

   Consider a face among $\{f_1, \dots, f_k\}$. In its sequence of S, perform the replacements

   $$e_1 \mapsto e_1^d, \qquad e_{k+1} \mapsto e_{k+1}^d, \qquad v_2 \mapsto v_2^d.$$

   This produces a valid sequences since

   $$\begin{aligned} \eta(e_1) = \{v_1, v_2\} &\qquad \text{implies} \qquad & \bar\eta(e_1^d) &= \{v_1, v_2^d\} \\ \eta(e_{k+1}) = \{v_2, v_3\} &\qquad \text{implies} \qquad & \bar\eta(e_{k+1}^d) &= \{v_2^d, v_3\} \\ \eta(e_j) = \{v, v_2\} &\qquad \text{implies} \qquad & \bar\eta(e_j) &= \{v, v_2^d\} \text{ for all } 2 \le j \le k. \end{aligned}$$

   A similar replacement works for the faces in $\{f_{k+1}, \dots, f_n\}$.

2. The vertices v_2^d and v_2^u are incident to the edge e^m. All other vertices are incident to an edge in E that also lies in $\bar E$.

3. The edge e^m is incident to the face g_1, the edge e_1^u lies in the face f_k, and so on.

Next, we show that $(\bar V, \bar E, \bar F, \bar\eta, \bar\varphi)$ actually is a simplicial complex, according to Definition 2.5.27.

1. For all edges not in $\{e_1^u, e_1^d, e_{k+1}^u, e_{k+1}^d, e^m\}$ there is nothing to show. For the edges in this set, it follows from inspection of $\bar\varphi$ that they are inner edges.

2. Clearly, only the vertices in $\{v_1, v_2^u, v_2^d, v_3\}$ have to be checked. The maximal umbrella of v_2^u is $(e_1^u, g_1, e^m, g_2, e_{k+1}^u, f_{k+1}, e_{k+2}, \dots, f_n, e_u)$. The maximal umbrella of v_2^d is $(e_1^d, f_1, e_2, \dots, e_k, f_k, e_{k+1}^d, g_2, e^m, g_1, e_1^d)$. In the maximal umbrella around v_1 in S, we replace the subsequence (f_1, e_1, f_n) by $(f_1, e_1^d, g_1, e_1^u, f_n)$. Analogously, for v_3 we replace (f_k, e_{k+1}, f_{k+1}) by $(f_k, e_{k+1}^d, g_2, e_{k+1}^u, f_{k+1})$. □

Remark 8.1.2. *Let S be a vertex–faithful simplicial surface. Then, each vertex split of S is also vertex–faithful.*

Proof. We use the notation from Definition 8.1.1. Clearly, $\bar\eta$ is injective if η is injective. Next, we need to consider $\bar\eta \mho \bar\varphi$:

$$\bar\eta \mho \bar\varphi : \bar F \to \mathrm{Pot}_3(\bar V) \qquad x \mapsto \begin{cases} \{v_1, v_2^u, v_2^d\} & x = g_1 \\ \{v_3, v_2^u, v_2^d\} & x = g_2 \\ \varphi(x)\backslash\{v_2\} \cup \{v_2^d\} & x \in \{f_1, \dots, f_k\} \\ \varphi(x)\backslash\{v_2\} \cup \{v_2^u\} & x \in \{f_{k+1}, \dots, f_n\} \\ \varphi(x) & \text{otherwise} \end{cases}$$

Clearly, this map is injective if $\eta \mho \varphi$ is. □

Vertex splits are just one kind of local modifications. While we do not define local modifications in detail, we can give a rough intuition about them. We can interpret a vertex split as follows:

1. Start with a simplicial surface.

2. Take a subsurface whose topological realisation is a disc.

3. Replace this subsurface by a different subsurface, whose topological realisation is also a disc.

Here, the replacing subsurface is crucial. If we choose a different replacement, we obtain a different local modification. Right now, we have no need to define local modifications in that level of generality, so we continue to analyse vertex splits.

To approach this problem, we ask whether we obtain more fine–grained information if the possible triangulations are restricted, e. g. by restricting the possible degrees.

Definition 8.1.3. *A simplicial surface is called* ***456–surface*** *if*

1. *The degree of each inner vertex lies in* $\{4, 5, 6\}$.

2. *The degree of each boundary vertex is smaller than 6.*

At this point, it becomes interesting to know how vertex splits change the vertex degrees.

Remark 8.1.4. *Let* S *be a vertex–faithful simplicial surface and* e_1, e_{k+1} *be two edges with* $\eta(e_1) = \{v_1, v_2\}$ *and* $\eta(e_{k+1}) = \{v_2, v_3\}$. *In the notation of definition 8.1.1 the vertex split has the following degrees:*

- *The degree of* v_1 *and* v_3 *increased by 1.*
- *The degree of* v_2^u *is* $n - k + 2$.
- *The degree of* v_2^d *is* $k + 2$.

Proof. The claims follow from the definition of $\bar{\varphi}$. □

Since 456–surfaces can only have the degrees 4, 5, and 6, the possible vertex splits are restricted.

Corollary 8.1.5. *Let* S *be a vertex–faithful 456–surface and* e, f *be two edges with* $\eta(e) = \{a, b\}$ *and* $\eta(f) = \{b, c\}$. *Then, the vertex split is a 456–surface if and only if*

1. *The degree of* a *and* c *is not 6.*

2. *The edges* e *and* f *are not incident to a common face.*

Proof. We use the notation from Definition 8.1.1 and the degree characterisation from Remark 8.1.4. We have $n \in \{4, 5, 6\}$ and $1 \leq k < n$. Therefore, we only have to prevent $k \in \{1, n - 1\}$. These cases appear if and only if e and f are incident to a common face. □

In particular, the only possible vertex splits happen between two singularities (recall Definition 4.1.1) with distance two.

This implies the analysis of a simplicial surface can proceed locally. If it is impossible that two vertices can interact (even indirectly) by vertex splits, then we can analyse each of them individually. To analyse this, we introduce *neighbours*, the set of adjacent vertices of a given vertex.

Definition 8.1.6. *Let (V, E, F, η, φ) be a polygonal complex and $v \in V$ a vertex. The* ***neighbours*** *of v are*

$$N(v) := \{w \in V \mid \exists e \in E \text{ with } \eta(e) = \{v, w\}\}.$$

With the neighbour–concept, we can reformulate Corollary 8.1.5.

Corollary 8.1.7. *Let (V, E, F, η, φ) be a vertex–faithful 456–surface and a, b two singularities. Then, there can be a vertex split between them if and only if*

1. *$a \neq b$.*
2. *There is no edge $e \in E$ with $\eta(e) = \{a, b\}$.*
3. *$N(a) \cap N(b) \neq \emptyset$.*

Since vertex splits can only happen between singularities that are quite close to each other, it is natural to restrict the surface to these neighbourhoods.

Definition 8.1.8. *Let $S = (V, E, F, \eta, \varphi)$ be a vertex–faithful 456–surface. A* ***defect neighbourhood*** *is a minimal subset $N \subseteq V$ such that for all singularities v holds:*

1. *$v \in N$ implies $N(v) \subseteq N$.*
2. *If there is a $w \in N$ and an $e \in E$ with $\eta(e) = \{v, w\}$, then $v \in N$.*

Since defect neighbourhoods should simplify the analysis, they should be disjoint. The next lemma shows that this is true.

Lemma 8.1.9. *Let S be a vertex–faithful 456–surface and N be a defect neighbourhood. For any singularity $v \in N$ holds: The minimal (with respect to inclusion) defect neighbourhood that contains v is equal to N.*

Proof. The minimal defect neighbourhood containing v is unique:

- v and $N(v)$ have to be contained.
- Whenever there are two defects with distance 2, one lying in the constructed neighbourhood and one outside, both (together with their neighbours) are added.

This defines a construction graph: The vertices are the singularities in N and two singularities are connected by an edge if their distance is at most 2. The above construction of the defect neighbourhood follows a connected component of this graph. Since this is independent from the start, the claim follows. □

Corollary 8.1.10. *Let S be a vertex–faithful 456–surface and N_1, N_2 two defect neighbourhoods. Then $N_1 = N_2$ or $N_1 \cap N_2 = \emptyset$.*

Proof. Assume $v \in N_1 \cap N_2$. If v is a singularity, Lemma 8.1.9 shows $N_1 = N_2$.

Otherwise, v is non–singular. Then, there exist singularities $s_1 \in N_1$ and $s_2 \in N_2$, together with edges e_1 and e_2 satisfying $\eta(e_i) = \{v, s_i\}$. By Definition 8.1.8, this implies $s_2 \in N_1$ and $s_1 \in N_2$. An application of Lemma 8.1.9 shows $N_1 = N_2$. □

Now we see the following structure:

- The vertex splits operate within defect neighbourhoods.
- The whole surface is determined by its defect neighbourhoods and the connections between them.

This gives two research questions:

- What happens within defect neighbourhoods? How can the neighbourhood be modified by vertex splits?
- How can the surface be recombined from its defect neighbourhoods?

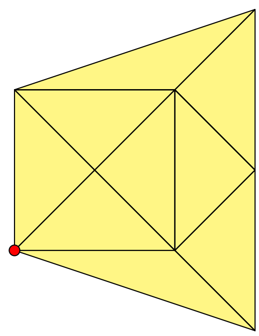
In this chapter, we give a partial solution to the first question.

8.2 Infinite regular extension

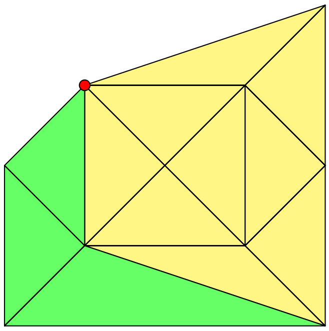
In the previous Section 8.1, we introduced the operation of vertex splitting and we analysed the local behaviour of this operation in the case of restricted vertex degrees.

In this section, we consider simplicial surfaces with exactly one boundary component. We can interpret such a surface as the simplest case of defect neighbourhoods from Section 8.1. We then construct a larger surface from it by extending its boundary with the methods from Chapter 6. This culminates in a general construction procedure for an infinite extension of the starting surface.

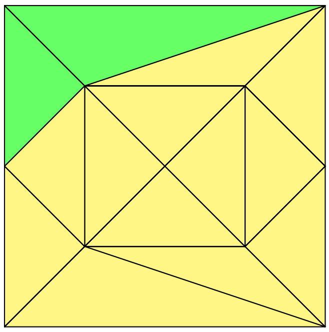
The main goal of this section is to define the infinite extension process and to prove the uniqueness of the resulting surface. To build an intuition for these extensions, consider the following simplicial surface:



If we interpret this surface as a defect neighbourhood, every vertex on the boundary has degree 6 in the larger surface. Therefore, an extension at the marked vertex would need to add three faces:



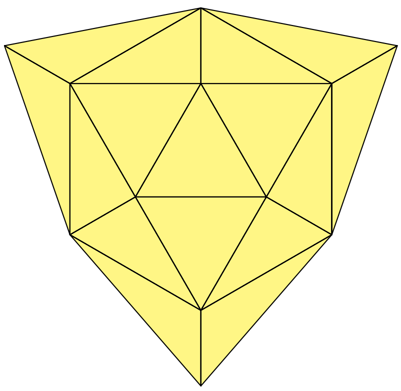
Now, the extension along the boundary vertex next to it only adds two faces:



Naturally, this construction can be repeated. So we could ask whether we can continue this construction indefinitely and, if so, whether the resulting surface is unique.

If we allow the extensions to become more complicated than discs or planes, they do not have to be unique, as the following example shows.

Example 8.2.1. *Consider the simplicial surface*



It can be extended into an infinite plane (which we show later in Theorem 8.3.9), but it can also be extended into a projective plane (with a finite number of triangles) by identifying opposite vertices.

Thus, our goal in this section is to define a construction procedure that gives a unique infinite extension, together with a criterion that tells us whether this is possible.

A naive approach would be to extend a random boundary vertex at each step. Unfortunately, this naive strategy has some pitfalls.

1. At each step in the construction, we should have a valid partial surface. If we notice later that some vertices assumed to be different are actually identical, this might wreak havoc on the construction because of conflicting requirements.

   We will conceptualise this by the requirement that there is a twilight morphism from the partial surface to the extended surface.

2. If we are not careful in our choice of boundary vertices, we might not be able to avoid to construct different vertices that turn out to be identical later on. For example, start with any disc triangulation. Then construct a sequence of triangles on both sides, like tentacles growing from the original disc. Some of the triangles on these tentacles might be shown to be equal later on but this is almost impossible to tell beforehand.

   To avoid this situation, we will essentially "ban tentacles" by requiring that the boundary of all partial surfaces is homogeneous enough to avoid these problems.

The construction relies heavily on the theory of cyclic sequences that we developed in Subsection 3.4.2. We restrict our attention to simplicial surfaces with exactly one connected boundary component (SB–surfaces from Definition 3.4.17). Theoretically, we could extend this construction to an arbitrary number of components (by applying the construction to each component individually), but we do not pursue this direction in this thesis.

We can avoid "too concave" boundaries by requiring the SB–surfaces to be growth–controlled (compare Definition 4.2.4).

With these definitions, we can start with the construction. Since the infinite regular extension is constructed "in the limit", we define finite extensions first.

Definition 8.2.2. *Let* $(S, \widehat{\deg}_S)$ *be a growth–controlled extended SB–surface. Then,* $(T, \widehat{\deg}_T, (v))$ *is a* ***vertex extension of*** S *if*

1. $(T, \widehat{\deg}_T)$ *is a growth–controlled extended SB–surface.*
2. v *is a boundary vertex of* T*.*
3. $T^{-v} = S$ *and* $\widehat{\deg}_T^{-v} = \widehat{\deg}_S$*.*

Furthermore, $(T, \widehat{\deg}_T, (v_1, \dots, v_k))$ *is a* ***vertex extension of*** S *if there is a vertex extension* $(U, \widehat{\deg}_U, (v_1, \dots, v_{k-1}))$ *of* S*, such that* $(T, \widehat{\deg}_T, (v_k))$ *is a vertex extension of* $(U, \widehat{\deg}_U)$*.*

With finite extensions in place, we can give the set of all vertex extensions a categorical structure, in order to identify the infinite regular extension as a limit in this category.

Definition 8.2.3. *Let $(S, \widehat{\deg}_S)$ be a growth–controlled extended SB–surface. Its* ***regular extension category*** *is defined as follows:*

- *The objects are vertex extensions $(T, \widehat{\deg}_T, (v_1, \dots, v_k))$ of S such that the induced extended twilight morphism $\mu_T : (S, \widehat{\deg}_S) \to (T, \widehat{\deg}_T)$ is hexagonal.*
- *The morphisms between $(T, \widehat{\deg}_T, (v_1, \dots, v_k))$ and $(U, \widehat{\deg}_U, (w_1, \dots, w_l))$ are extended twilight morphisms $\psi : (T, \widehat{\deg}_T) \to (U, \widehat{\deg}_U)$ such that $\psi \circ \mu_T = \mu_U$.*

If there is a strongly connected, vertex–faithful, closed simplicial surface S^∞ such that

1. *There exists a twilight morphism $\rho : S \to S^\infty$*
2. *For each object $(T, \widehat{\deg}_T, \mu)$ there is a twilight morphism $\rho_T : T \to S^\infty$ such that $\rho_T \circ \mu = \rho$,*

then S^∞ is called an ***infinite regular extension*** *of S and denoted by $\varprojlim S$.*

Ideally, we could show that an infinite regular extension always exists. Unfortunately, showing existence in general is quite difficult (it is much easier if we are restricted to special cases), so we postpone this until Section 8.3.

In this section, we prove the uniqueness of the infinite regular extension. To do so, we start with classifying the different possible vertex extensions with one vertex.

Remark 8.2.4. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface and v a boundary vertex of S. If $(S^{-v}, \widehat{\deg}^{-v})$ is a growth–controlled extended SB–surface, $\deg_S(v) \leq 3$.*

Proof. If we consider the construction of $(S^{-v}, \widehat{\deg}^{-v})$ from Lemma 6.1.8, we observe that the $\widehat{\deg}^{-v}$–sequence contains $\deg(v) - 1$ many 2's. Since there can be at most three 2's next to each other (by growth–control), the degree of v is at most 3. □

Although the surface S can have a very complicated topology, the constructed vertex extensions are pretty simple. They can even be mapped into the plane.

Definition 8.2.5. *Let $(S, \widehat{\deg})$ be an extended SB–surface and $(T, \widehat{\deg}_T, (v_1, \dots, v_k))$ a vertex–extension with twilight morphism $\mu : S \to T$. Let*

- *V be the set of vertices in T that do not lie in the image of μ or are the image of a boundary vertex of S.*
- *E be the set of edges in T that do not lie in the image of μ or are the image of a boundary edge in S.*
- *F be the set of faces in T that do not lie in the image of μ.*
- *$\eta : E \to \mathrm{Pot}_2(V)$ be the restriction of the map η from T.*
- *$\varphi : F \to \mathrm{Pot}_3(E)$ be the restriction of the map φ from T.*

A ***continuous embedding of*** $T \backslash S$ *consists of*

1. *An injective map* $\iota_V : V \to \mathbb{R}^2$,
2. *A map* ι_E *that associates to each* $e \in E$ *a continuous path* $p : [0,1] \to \mathbb{R}^2$ *such that* $\{p(0), p(1)\} = \{\iota_V(v) \mid v \in \eta(e)\}$,

fulfilling

1. *The paths* $\iota_E(e)$ *do not intersect, except at the end–points (and there, only if and only if the corresponding edges have a common vertex).*
2. *For each* $f \in F$ *the path* $\iota_E(e)$ *for* $e \in \varphi(e)$ *forms a* `Jordan`*–curve such that the bounded component of the complement contains no more elements from* ι_V *or* ι_E*.*
3. *The boundary* ∂S *is mapped to a* `Jordan`*–curve whose bounded component contains no elements from* ι_V *or* ι_E*.*

Lemma 8.2.6. *Any vertex–extension of* $(S, \widehat{\deg})$ *has a continuous embedding.*

Proof. We map ∂S to the circle of radius 1. Then, we construct the remaining map inductively.

We always start from the following position: The boundary of ∂S is mapped to a `Jordan`–curve in $\mathbb{R}^2$ and the unbounded component of the complement is empty, i.. it does neither contain a $\iota_V(v)$ for $v \in V$, nor a point on a path $\iota_E(e)$ for $e \in E$. To construct the embedding of S^{+v}, we proceed as follows:

1. Pick any vertex in the unbounded component as the image $\iota_V(v)$
2. Pick a face f incident to v. We can find two paths from $\iota_V(v)$ to the corresponding end–points. These paths are the paths $\iota_E(e)$ for the edges $e \in E$ that are incident to both v and f.

   We extend ∂S by these paths into a `Jordan`–curve such that the unbounded complement remains empty.
3. Pick a face adjacent to one already constructed. Then, two edges of this face already have an image under ι_E. Thus, the vertices incident to the final edge lie on a `Jordan`–curve whose unbounded complement component is empty.

   We pick any path in that component connecting the two vertices as the image of the final edge. We can extend the previous `Jordan`–curve to a new one whose unbounded complement component is empty.

   We repeat this process until all faces incident to v are constructed.

In this fashion, we can construct an arbitrary extension. □

Remark 8.2.7. *The only boundary edges of the continuous embedding are the images of* ∂S *and* ∂T*.*

We apply the continuous embedding to construct closed paths within the vertex extensions.

Lemma 8.2.8. *Let $(S, \widehat{\deg})$ be an extended SB–surface and $(T, \widehat{\deg}_T, (u_1, \dots, u_m))$ an object of the regular extension category, with hexagonal extended twilight morphism $(\mu_V, \mu_E, \mu_F) : (S, \widehat{\deg}) \to (T, \widehat{\deg}_T)$.*

Let $v_1, v_2 \in \partial S$ and assume there is a non–intersecting path

$$(\mu_V(v_1), w_1, w_2, \dots, w_k, \mu_V(v_2))$$

in T such that w_i does not lie in the image of μ_V (for all $1 \le i \le k$). Then, there exists a path $(v_1, s_1, \dots, s_n, v_2)$ in ∂S, such that

$$(\mu_V(v_1), \mu_V(s_1), \dots, \mu_V(s_n), \mu_V(v_2), w_k, \dots, w_1, \mu_V(v_1))$$

is a cyclic intersection–free path in T, whose inner vertices all have degree 6 and with $\deg(s_i) = \widehat{\deg}_S(s_i)$.

Proof. By Lemma 8.2.6, there is a continuous embedding. The path

$$(\mu_V(v_1), w_1, w_2, \dots, w_k, \mu_V(v_2))$$

becomes a continuous path between $\iota_V(v_1)$ and $\iota_V(v_2)$ (with notation from Definition 8.2.5).

Consider the paths on ∂S between $\iota_V(v_1)$ and $\iota_V(v_2)$. Both of these can be combined with the path in T to obtain a `Jordan`–curve that bounds a disc. By [69, Corollary 1.2], these three paths separate the plane into three faces, whose boundaries are formed of two paths each. By Definition 8.2.5, one of these bounded faces is formed from both paths on ∂S. Thus, the other bounded face has the path T and exactly one path in ∂S as boundary.

All vertices within this `Jordan`–curve have degree 6 by construction of the regular extension category. □

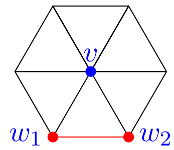
Next, we prove a technical lemma that is crucial to prove the uniqueness of the infinite regular extension.

Lemma 8.2.9. *Let $(S, \widehat{\deg}_S)$ be a growth–controlled extended SB–surface, $(T, \widehat{\deg}_T)$ an extended vertex–faithful simplicial surface, and $\mu = (\mu_V, \mu_E, \mu_F) : (S, \widehat{\deg}_S) \to (T, \widehat{\deg}_T)$ an injective extended hexagonal polygonal morphism.*

Let v be a regular vertex of T that does not lie in the image of μ_V. Let w_1 and w_2 be two vertices from the link $\mathrm{Lk}_T(v)$ that lie in the image of μ_V. Then, the image of μ contains a vertex–edge–path from w_1 to w_2 that has at most length 3.

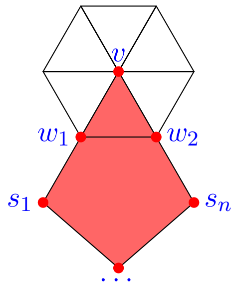
Proof. Since v is a regular vertex, its maximal umbrella is contained in a hexagon. We prove the claim by contradiction and consider the three possible relative positions of w_1 and w_2.

1. w_1 and w_2 have distance 1 in $\mathrm{Lk}_T(v)$.



If the edge between w_1 and w_2 in $\mathrm{Lk}_T(v)$ lies in the image of μ_E, the claim is true.

Otherwise, we can combine the vertex path (w_1, v, w_2) with $(w_1, s_1, \ldots, s_n, w_2)$, the image of a boundary path in S, to obtain a closed path in T that only encircles vertices with degree 6 (μ is hexagonal).



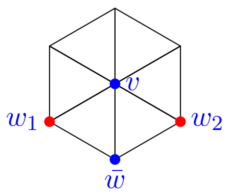
We compute the inner degrees of this polygon. By construction, $\deg(s_i) = \widehat{\deg}_S(s_i)$ and $\deg(v) = 1$. From Lemma 4.1.5, we obtain

$$6 = \sum_{i=1}^{n}(3 - \widehat{\deg}_S(s_i)) + (3 - 1) + (3 - \deg(w_1)) + (3 - \deg(w_2)),$$

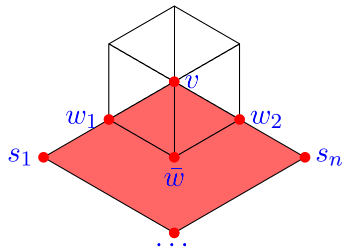
which can be rewritten as $\sum_{i=1}^{n}(3 - \widehat{\deg}_S(s_i)) = \deg(w_1) + \deg(w_2) - 2$.

Since the edge between w_1 and w_2 is not part of the path, we have $\deg(w_i) > 1$. If $\deg(w_1) = \deg(w_2) = 2$, the path would be (w_1, s_1, w_2, v, w_1) with $\widehat{\deg}_S(s_1) = 1$, which is impossible since S is growth–controlled (compare Definition 4.2.4). Thus, $\deg(w_1) + \deg(w_2) \geq 5$, implying $\sum_{i=1}^{n}(3 - \widehat{\deg}_S(s_i)) \geq 3$, which is a contradiction to S being growth–controlled.

2. w_1 and w_2 have distance 2 in $\mathrm{Lk}_T(v)$:



Since w_1 and w_2 are both images of boundary vertices in S, there is a path in the boundary of S, whose image is the path $(w_1, s_1, s_2, \ldots, s_n, w_2)$, such that $(w_1, s_1, \ldots, s_n, w_2, v, w_1)$ is a closed path in T, where all encircled vertices have degree 6.



With Lemma 4.1.5, we obtain like in the previous case

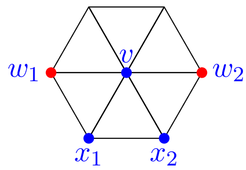
$$6 = \sum_{i=1}^{n}(3 - \widehat{\deg}_S(s_i)) + (3-2) + (3 - \deg(w_1)) + (3 - \deg(w_2)).$$

This implies $\sum_{i=1}^{n}(3 - \widehat{\deg}_S(s_i)) = \deg(w_1) + \deg(w_2) - 1$.

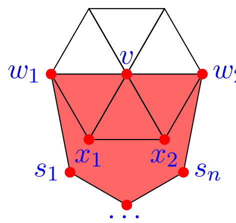
Since S is growth–controlled, this sum is at most 2. Thus, $\deg(w_1) + \deg(w_2) \leq 3$. If both degrees are 1, S contains the boundary path $(w_1, \bar{w}, w_2)$.

If $\deg(w_1) = 2$ and $\deg(w_2) = 1$, then we have the path $(w_1, s_1, \bar{w}, w_2)$ with $\widehat{\deg}_S(s_1) = 1$, contradicting S being growth–controlled.

3. w_1 and w_2 have distance 3 in $\mathrm{Lk}_T(v)$:



Like in the previous cases, a boundary path in S is mapped to $(w_1, s_1, \dots, s_n, w_2)$ by μ, such that $(w_1, s_1, \dots, s_n, w_2, v, w_1)$ is a closed path in T that only surrounds vertices of degree 6.



From Lemma 4.1.5 we obtain (like in the previous cases)

$$6 = \sum_{i=1}^{n}(3 - \widehat{\deg}_S(s_i)) + (3-3) + (3 - \deg(w_1)) + (3 - \deg(w_2)).$$

This gives $\sum_{i=1}^{n}(3 - \widehat{\deg}_S(s_i)) = \deg(w_1) + \deg(w_2)$. Since S is growth–controlled, this sum is at most 2. Since $\deg(w_i) \geq 1$, this implies $\deg(w_i) = 1$.

Then, the two vertices x_1 and x_2 have to be contained in the image of μ. By the first case of this lemma, this implies that the edge connecting them also has to be contained in the image of S. Therefore, the path (w_1, x_1, x_2, w_2) lies in the image of μ. $\square$

At this point, we can formulate the uniqueness lemma (for the finite case).

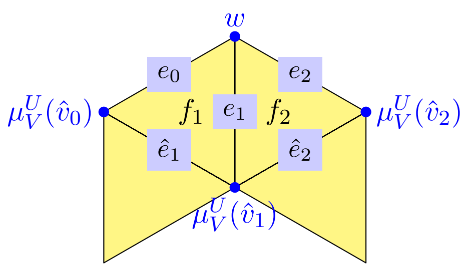
Lemma 8.2.10. *Let $(S, \widehat{\deg}_S)$ be a growth–controlled extended SB–surface.*

Let $(T, \widehat{\deg}_T, (v_1, \ldots, v_k))$ and $(U, \widehat{\deg}_U, (w))$ be objects in the regular extension category of $(S, \widehat{\deg}_S)$ with hexagonal extended twilight morphisms

$$\mu^T : (S, \widehat{\deg}_S) \to (T, \widehat{\deg}_T) \qquad\qquad \mu^U : (S, \widehat{\deg}_S) \to (U, \widehat{\deg}_U).$$

If $\mu_E^T \circ (\mu_E^U)^{-1}(e)$ is an inner edge of T for every edge in $\mathrm{Lk}_U(w)$, there is a unique hexagonal extended twilight morphism $\psi : (U, \widehat{\deg}_U) \to (T, \widehat{\deg}_T)$ with $\psi \circ \mu^U = \mu^T$.

Proof. Denote $T = (V^T, E^T, F^T, \eta^T, \varphi^T)$ and $U = (V^U, E^U, F^U, \eta^U, \varphi^U)$. We have an extended twilight morphism $(\mu_V^T, \mu_E^T, \mu_F^T) : S \to T$ and $(\mu_V^U, \mu_E^U, \mu_F^U) : S \to U$. Let the maximal umbrella of w in U be $(e_0, f_1, e_1, \ldots, f_n, e_n)$ with $n \leq 3$ by Remark 8.3.1. Assume $\varphi^U(f_i) = \{e_{i-1}, e_i, \hat{e}_i\}$, so $\hat{e}_i$ are the edges of $\mathrm{Lk}(w)$. Let $\hat{v}_i \in V^S$ such that $\eta^U(\hat{e}_i) = \{w, \mu_V^U(\hat{v}_i)\}$.

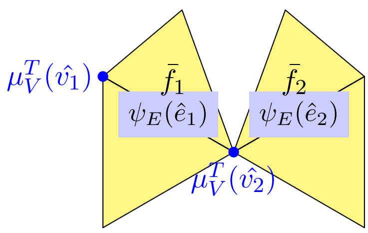


To fulfil $\psi \circ \mu^U = \mu^T$, the morphism ψ has to have the following partial definition:

$$\begin{aligned}
\psi_V :& \quad V^U \backslash \{w\} \to V^T & v &\mapsto \mu_V^T \circ (\mu_V^U)^{-1}(v) \\
\psi_E :& \quad E^U \backslash \{e_0, e_1, \ldots, e_n\} \to E^T & e &\mapsto \mu_E^T \circ (\mu_E^U)^{-1}(e) \\
\psi_F :& \quad F^U \backslash \{f_1, \ldots, f_n\} \to F^T & f &\mapsto \mu_F^T \circ (\mu_F^U)^{-1}(f)
\end{aligned}$$

We have to show that ψ can be uniquely (and consistently) defined for the missing values.

This is easy for the faces $\{f_1, \ldots, f_n\}$. They are incident to the edges $\hat{e}_i \in E^U$. Since $\mu_E^T \circ (\mu_E^U)^{-1}(\hat{e}_i)$ are inner edges in T, there are unique faces $\bar{f}_i$ to which we can map f_i.



We have to show that $\bar{f}_{i-1}$ and $\bar{f}_i$ are incident to the same edge. To do so, consider the vertex $\mu_V^T(\hat{v}_i)$ between them. Since both μ^U and μ^T are extended twilight morphisms, we have

$$\deg_T(\mu_V^T(\hat{v}_i)) + \widehat{\deg}_T(\mu_V^T(\hat{v}_i)) = \deg_S(\hat{v}_i) + \widehat{\deg}_S(\hat{v}_i) = \deg_U(\mu_V^U(\hat{v}_i)).$$

We know that $\deg_U(\mu_V^U(\hat{v}_i)) = \deg_S(\hat{v}_i) + 2$, so we can conclude that $\mu_V^T(\hat{v}_i)$ is an inner vertex of T and that $\bar{f}_{i-1}$ and $\bar{f}_i$ are incident to the same edge. This defines ψ_E uniquely for $\{e_0, e_1, \ldots, e_n\}$. Furthermore, all faces $\bar{f}_i$ (for $1 \leq i \leq n$) are incident to a common vertex, which has to be $\psi_V(w)$.

It remains to prove that the polygonal morphism ψ is a polygonal shadow morphism, according to Definition 2.7.8. We start by establishing that ψ_V is injective.

- Let $v_1, v_2 \in V^U \backslash \{w\}$ with $\psi_V(v_1) = \psi_V(v_2)$. This is equivalent to

  $$\mu_V^T \circ (\mu_V^U)^{-1}(v_1) = \mu_V^T \circ (\mu_V^U)^{-1}(v_2).$$

  Since μ_V^T is injective (by Remark 2.3.4 and Lemma 2.7.9), this implies $(\mu_V^U)^{-1}(v_1) = (\mu_V^U)^{-1}(v_2)$, which implies $v_1 = v_2$ in turn.

- Let $v \in V^U \backslash \{w\}$ and assume $\psi_V(v) = \psi_V(w)$. Then, $(\mu_V^U)^{-1}(v) \in V^S$. In addition,

  $$\{\psi_V(w), \mu_V^T(\hat{v}_1), \mu_V^T(\hat{v}_2)\} = (\eta^T \between \varphi^T)(f)$$

  for a face $f \in F^T$. Since μ_V^T is an injective polygonal shadow morphism,

  $$\{(\mu_V^U)^{-1}(v), \hat{v}_1, \hat{v}_2\} = (\eta^S \between \varphi^S)(g)$$

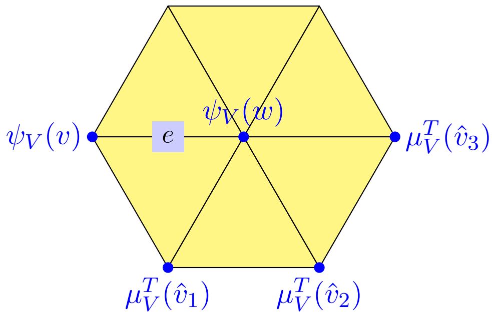
  for a face $g \in F^S$. Since the edge $(\mu_E^U)^{-1}(\hat{e}_1)$ is a boundary edge in S, the face g is the only incident face. Since $\mu_E^T \circ (\mu_E^U)^{-1}(\hat{e}_1)$ is an inner edge in T, it is incident to two faces, $\mu_F^T(g)$ and $\bar{f}_1$. But since T is vertex–faithful, $\psi_V(w) = \psi_V(v)$ implies $g = \bar{f}_1$, which is a contradiction.

Let $X \in \mathrm{Pot}(V^U)$ and consider $Y = \{\psi^V(y) \mid y \in X\}$. Since ψ^V is injective, $|X| = |Y|$. Thus, only the cases for $|X| \in \{1, 2, 3\}$ are relevant.

1. $|Y| = 1$. Then, $X = \{v\}$ for some $v \in V^U$, since ψ^V is injective.
2. $|Y| = 2$. If $w \notin X = \{v_1, v_2\}$, we have $Y = \{\mu_V^T \circ (\mu_V^U)^{-1}(v_1), \mu_V^T \circ (\mu_V^U)^{-1}(v_2)\}$. Since μ^T is a polygonal shadow morphism, there is an edge $e_S \in E^S$ with $\eta^S(e_S) = \{(\mu_V^U)^{-1}(v_1), (\mu_V^U)^{-1}(v_2)\}$. Since μ_U is a polygonal morphism, there is an edge $e_T \in E^T$ with $\eta^T(e_T) = \{v_1, v_2\}$.

   Otherwise, $X = \{w, v\}$ for some $v \in V^U \backslash \{w\}$. If there is an edge in $e \in E^T$ with $\eta^T(e) = Y$, consider the umbrella around $\psi_V(w)$. Since μ^T is hexagonal, this umbrella lies in a hexagon.

   If $\psi_V(v) \notin \{\mu_V^T(\hat{v}_1), \ldots, \mu_V^T(\hat{v}_n)\}$, the vertices $\psi_V(v)$ and $\{\mu_V^T(\hat{v}_1), \ldots, \mu_V^T(\hat{v}_n)\}$ all lie on the boundary of the hexagon.



We apply Lemma 8.2.9 to ψ. It shows that there is a vertex–edge–path along the boundary of the hexagon that connects $\psi_V(v)$ to $\mu_V^T(\hat{v}_i)$ and lies in the image of ψ. Consider the pre–image of this path in U. All edges of this path lie in $\mathrm{Lk}_U(w)$.

By Remark 8.3.1, the link contains at most 4 vertices and 3 edges. Thus, v is adjacent to $\mu_V^U(\hat{v}_1)$ or $\mu_V^U(\hat{v}_n)$. But this implies $\widehat{\deg}_U(\mu_V^U(\hat{v}_1)) = 1$, contradicting the growth–control of U.

3. $|Y| = 3$. We argue similar to the second case that $X = \{w, v_1, v_2\}$ for some $v_1, v_2 \in V^U \backslash \{w\}$ is the only case that is yet to prove. If Y consists of the vertices of a face in T, all of its edges also lie in T. By the already proven second case, this implies that $\{v_1, v_2\}$, $\{w, v_1\}$, and $\{w, v_2\}$ are the vertices of certain edges in U.

   Thus, $v_1, v_2 \in \{\mu_V^U(\hat{v}_0), \mu_V^U(\hat{v}_1), \ldots, \mu_V^U(\hat{v}_n)\}$. Up to symmetry, we have these two cases:

   - If $v_1 = \mu_V^U(\hat{v}_i)$ with $0 < i < n$, we have $v_2 \in \{\mu_V^U(\hat{v}_{i-1}), \mu_V^U(\hat{v}_{i+1})\}$, otherwise the edge $\hat{e}_i$ would be ramified.
   - If $v_1 = \mu_V^U(\hat{v}_0)$ and $v_2 = \mu_V^U(\hat{v}_n)$, we deduce that w is an inner vertex of U with $\deg_U(w) = n + 1$.

     By Remark 8.3.1, $n \leq 3$. But $(U, \widehat{\deg}_U, (w))$ is an object of the regular extension category, which implies that w is a regular vertex. This contradiction proves the impossibility of this case. $\square$

With Lemma 8.2.10 in hand, we can show the uniqueness of the infinite regular extension.

Theorem 8.2.11. *Let $(S, \widehat{\deg}_S)$ be a growth–controlled extended SB–surface. Let $T^k = (T_k, \widehat{\deg}_{T_k}, V_k)$ be objects from the regular extension category, such that*

1. *There are morphisms $\mu_k : T^k \to T^{k+1}$, with the stipulation that $T^0 := (S, \widehat{\deg}_S, ())$.*
2. *For any $k \in \mathbb{N}$, there is a $K \in \mathbb{N}$ such that all boundary edges of T_k are inner edges in T_K.*
3. *There exists a strongly connected, vertex–faithful, closed simplicial surface T_∞ with extended twilight morphisms from $(T_k, \widehat{\deg}_{T_k})$ to (T_∞) that commute with all μ_k.*

Then, T_∞ is the infinite regular extension of $(S, \widehat{\deg}_S)$.

Proof. Let $(T, \widehat{\deg}_T, (v_1, \dots, v_l))$ be any object from the regular extension category. This gives the sequence

$$\begin{aligned}(S, \widehat{\deg}_S, ()) &\to ((((T^{-v_l})^{-v_{l-1}})^{\cdots})^{-v_2}, \widehat{\deg}_{T_1}, (v_1)) \\ &\to \dots \\ &\to (T^{-v_l}, \widehat{\deg}_{T_{l-1}}, (v_1, \dots, v_{l-1})) \\ &\to (T, \widehat{\deg}_T, (v_1, \dots, v_l)).\end{aligned}$$

By assumption, there is a $K \in \mathbb{N}$ such that all boundary edges of S are inner edges in T_K. We apply Lemma 8.2.10 to construct a morphism from the second sequence–term to T^K.

We repeat this argument, which leads to a morphism $T \to T^L$ for some $L \in \mathbb{N}$. From there, it is easy to see that we end up in T_∞. □

8.3 Construction of infinite regular extension

In the previous Section 8.2, we introduced the concept of infinite regular extension and showed its uniqueness in Theorem 8.2.11.

In this section, we turn to the actual construction of infinite regular extensions in several cases. Since the regular extension category might not always exist, we start by developing conditions that are necessary for existence.

Remark 8.3.1. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface. Then, the total extended defect of $(S, \widehat{\deg})$ is at most 6χ.*

Proof. From Corollary 4.2.9 we obtain

$$\sum_{v \in V_B} (3 - \widehat{\deg}(v)) = \sum_{v \in V} \widehat{\operatorname{def}}(v) - 6\chi.$$

Since $(S, \widehat{\deg})$ is growth–controlled, the sum on the left is at most 0. Thus,

$$\sum_{v \in V} \widehat{\operatorname{def}}(v) \leq 6\chi. \qquad \square$$

A crucial property in the regular extension category is *growth–controlled*. Since this property only depends on the boundary of the surface, we define the *boundary defect* to organise the behaviour of the vertex–extensions.

Definition 8.3.2. *Let $(S, \widehat{\deg})$ be an extended SB–surface. Its* ***boundary defect*** *is*

$$\sum_{v \in V_B} (3 - \widehat{\deg}(v)).$$

We employ the following construction strategy: First, we manually extend the triangulation until we obtain a standard form for the boundary. Then, we combine it with a generic infinite surface and use the uniqueness from Theorem 8.2.11 to validate this construction.

Lemma 8.3.3. *Let $(S, \widehat{\deg})$ be a growth–controlled, extended SB–surface. There is an object $(T, \widehat{\deg}_T, (v_1, \ldots, v_k))$ in the regular extension category such that all cyclic intervals of ∂T have defect–sum at most 1.*

In addition, $\widehat{\deg}_T$ does take the value 5 as most as often as $\widehat{\deg}$.

Proof. Recall the notation from Definition 3.4.11. Assume there is a cyclic interval C in ∂S with $d_{\widehat{\deg}}(C) = 2$. We will show that there is a cyclic subinterval $(v_0, v_1, \ldots, v_n)$ such that $\widehat{\deg}(v_0) = \widehat{\deg}(v_n) = 2$ and $\widehat{\deg}(v_i) = 3$ for all $1 \leq i < n$.

Assume to the contrary that there is no such subinterval. Consider the set $C_2 := \{c \in C \mid \widehat{\deg}(c) = 2\}$. Since $(S, \widehat{\deg})$ is growth–controlled, there is no $c \in C$ with $\widehat{\deg}(c) = 1$. Therefore, an element x with $\widehat{\deg}(x) > 3$ lies between any two elements of C_2. This implies $d_{\widehat{\deg}}(C) \leq 1$, in contradiction to $d_{\widehat{\deg}}(C) = 2$. Thus, there is at least one such interval.

Consider all cyclic intervals $(v_0, v_1, \ldots, v_n)$ with $\widehat{\deg}(v_0) = \widehat{\deg}(v_n) = 2$ and $\widehat{\deg}(v_i) = 3$ for all $1 \leq i < n$. Pick one where n is minimal. We distinguish two cases:

- If $n = 1$, the assumptions of Lemma 6.2.13 are satisfied. We replace $(S, \widehat{\deg})$ by the extension of Lemma 6.2.12, which reduces the length of the boundary by one.
- If $n > 1$, we have $\widehat{\deg}(v_{n-1}) = 3$. Consider the other vertex adjacent to v_n (call it w). If $\widehat{\deg}(w) = 2$, the defect–sum of $(v_0, \ldots, v_n, w)$ would be 3, in contradiction to the growth–control. Thus, $\widehat{\deg}(w) \geq 3$. Therefore, the assumptions of Lemma 6.2.9 are satisfied. We replace $(S, \widehat{\deg})$ be the extension of Lemma 6.2.8, which reduces the minimal length n by one.

We apply the second case until $n = 1$, and reduce the boundary length with the first case. Since the boundary length is finite, this process terminates after a finite number of steps.

Since we only used the extensions of Lemma 6.2.8 and Lemma 6.2.12, and both of them do not increase the number of vertices with value 5, the additional claim follows by induction. ∎

The remaining work depends on the value of the boundary defect.

8.3.1 Boundary defect 0

In this subsection, we perform the explicit construction of the infinite regular extension for extended SB–surfaces with boundary defect 0. Roughly, the extension is constructed from the SB–surface and an infinite cylinder. The possible shapes of this extensions correspond to the shapes of *nanotubes*, classified in [17].

Since we want to combine a generically defined cylinder with the SB–surface, we extend the surface along its boundary until the external degree sequence has a "nice" form. We start off by defining what we mean by "nice" form in the case of boundary defect 0.

Definition 8.3.4. *Let $(S, \widehat{\deg})$ be an extended SB–surface with boundary defect 0. Assume ∂S has the vertices (in order) $v_1, v_2, \dots, v_n$.*
*$\widehat{\deg}$ is a **staircase** if $\widehat{\deg}(v_i) = 4$ if and only if $\widehat{\deg}(v_{i+1}) = 2$ holds for all $i \in \mathbb{Z}/n\mathbb{Z}$.*
*We call $\widehat{\deg}$ an m–**staircase** (for $2m \leq n$) if*

1. *$d(2i-1) = 4$ for all $1 \leq i \leq m$.*
2. *$d(2i) = 2$ for all $1 \leq i \leq m$.*
3. *$d(j) = 3$ for all $2m < j \leq n$.*

Crucially, an extended SB–surface can be extended in such a way that its external degree sequence is an m–staircase. The value of m is actually unique, but we postpone the proof of this fact until Lemma 8.3.10.

Lemma 8.3.5. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0. There exists an element $(T, \widehat{\deg}_T, (w_1, \dots, w_k))$ in the regular extension category such that $\widehat{\deg}_T$ is an m–staircase for some m.*
The boundary length of T is at most as large as the boundary length of S.

Proof. By Lemma 8.3.3, there is a growth–controlled extended SB–surface $(U, \widehat{\deg}_U)$, where the defect–sum of every cyclic interval is at most 1.

Consider the maximal cyclic intervals in which $\widehat{\deg}_U$ does only take values larger than 2 (called *stretch intervals*). Between these maximal intervals are cyclic intervals where $\widehat{\deg}_U$ takes only the value 2 (called *glue intervals*). Since the defect–sum of all cyclic intervals is at most 1, all glue intervals have length 1. If any of the stretch intervals had defect–sum 0, extending it by the glue intervals on both sides would make its defect–sum 2. Thus, the stretch intervals have defect–sum at most -1. Since there are exactly as many stretch intervals as glue intervals, the defect–sum of each stretch interval is exactly -1 (the boundary defect is 0). Thus, each of the stretch intervals contains exactly one vertex v with $\widehat{\deg}_U(v) = 4$.

If all vertices have external degree 3, $\widehat{\deg}_U$ is a 0–staircase. Otherwise, we construct it manually. Without loss of generality, choose an enumeration $v_1, v_2, \dots, v_n$ of the vertices in ∂U, such that $\widehat{\deg}_U(v_1) = 4$. Then, consider the smallest i with $\widehat{\deg}_U(v_i) = 4$ and $\widehat{\deg}_U(v_{i+1}) \neq 2$. There is a minimal $j > i$ with $\widehat{\deg}_U(v_j) = 2$. We apply the extension from Lemma 6.2.8 to v_j to obtain a growth–controlled (Lemma 6.2.9) extended surface $(U^+, \widehat{\deg}^+)$ with boundary $(v_1, v_2, \dots, v_{j-1}, v, v_{j+1}, \dots, v_n)$, where $\widehat{\deg}^+(v_{j-1}) = 2$.

We repeat these extensions until $\widehat{\deg}_U$ is a staircase. Then, we choose an enumeration $(v_1, v_2, \dots, v_n)$ such that $\widehat{\deg}_U(v_1) = 2$ and $\widehat{\deg}_U(v_2) = 4$ (we flip the orientation of the previous enumeration). We apply the same argument as before to obtain an m–staircase. □

Lemma 8.3.5 shows that we can construct an extension whose external degree sequence is a staircase. Next, we define an infinite surface (a cylinder) that can be combined with it. To see a graphic representation of it, consider Example 8.3.7.

Definition 8.3.6. *The **hexagonal cylinder with waist–length** w **and offset** m (with $w \geq 3$ and $0 \leq m \leq \frac{w}{2}$) is the vertex–faithful simplicial surface (V, E, F, η, φ), with*

- $V = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq w, y \geq \max(0, m - \lceil\frac{x}{2}\rceil)\}/\sim$ *where* $\sim$ *is an equivalence relation with the equivalence classes* $\{(x, y)\}$ *for* $0 < x < w$ *and* $\{(w, y), (0, y+m)\}$ *for all* $y \geq 0$.
- $E = E_- \uplus E_| \uplus E_\backslash$, *with*

$$\begin{aligned}
E_- &= \{\{(x, y), (x + 1, y)\} \mid 0 \leq x < w, y \geq \max(0, m - \lceil\frac{x}{2}\rceil)\} \\
E_| &= \{\{(x, y), (x, y + 1)\} \mid 0 \leq x < w, y \geq \max(0, m - \lceil\frac{x}{2}\rceil)\} \\
E_\backslash &= \{\{(x, y), (x - 1, y + 1)\} \mid 0 < x \leq w, y \geq \max(0, m - \lceil\frac{x}{2}\rceil)\}.
\end{aligned}$$

- $F = F_+ \uplus F_-$, *with*

$$\begin{aligned}
F_+ &= \{\{(x, y), (x + 1, y), (x, y + 1)\} \mid 0 \leq x < w, y \geq \max(0, m - \lceil\frac{x}{2}\rceil)\} \\
F_- &= \{\{(x, y), (x, y + 1), (x - 1, y + 1)\} \mid 0 < x \leq w, y \geq \max(0, m - \lceil\frac{x}{2}\rceil)\}.
\end{aligned}$$

- $\eta : E \to \mathrm{Pot}_2(V)$ *is defined by* $\eta(e) = \{[v] \mid v \in e\}$, *where* $[v]$ *denotes the equivalence class of* v *with respect to* $\sim$.
- $\varphi : F \to \mathrm{Pot}_3(E)$ *is defined by* $\varphi(f) = \{e \in E \mid e \subseteq f\}$.

Well–defined. We have to show that (V, E, F, η, φ) defines a vertex–faithful simplicial surface. The map η is well–defined since $w > 1$. Clearly, φ is well–defined.

We use the characterisation from Lemma 2.7.5. For that, we have to construct the map $\eta \between \varphi : F \to \mathrm{Pot}_3(V)$ explicitly:

$$(\eta \between \varphi)(f) = \{[v] \mid v \in f\}$$

Then, we can check the individual conditions of the lemma.

1. Let $e_1, e_2 \in E$ with $\eta(e_1) = \eta(e_2)$. We distinguish three cases, depending on the sizes of the equivalence classes in $\eta(e_1)$.
   - If both equivalence classes have size 1, $e_1 = e_2$ follows immediately.
   - If both equivalence classes have size 2, there are $y_1, y_2 \in \mathbb{Z}$ with

     $$\eta(e_1) = \{\{(w, y_1), (0, y_1 + m)\}, \{(w, y_2), (0, y_2 + m)\}\}.$$

     Since $w > 1$m this is only possible if $|y_2 - y_1| = 1$, which uniquely determines e_1.

- Otherwise, there is an $(x, y) \in e_1 \cap e_2$. Since $w > 2$, either $x = 1$ or $x = w - 1$. But then, $e_1 = e_2$ follows easily.

With similar reasoning, $\eta \mho \varphi$ is injective.

2. Clearly, every vertex lies in an edge.
3. It is easy to see that every edge lies in a face.
4. It is also easy to see that each 2–element–subset of a face is an edge.

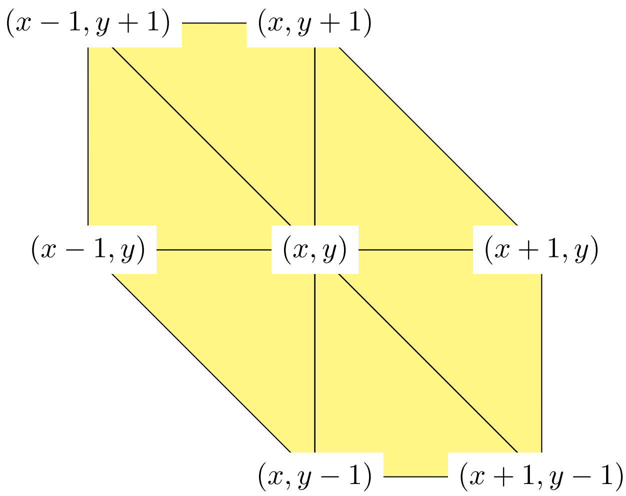
Thus, the cylinder is a vertex–faithful triangular complex.

To show that it is in fact a simplicial surface, we need to check the properties of Definition 2.5.27. It is easy to see that every edge is incident to at most two faces.

Consider the maximal umbrellas. We start with $(x, y) \in \mathbb{Z}^2$, where $0 < x < w$ and $y > \max(0, m - \lceil \frac{x}{2} \rceil)$. Then, the following edges are incident to (x, y):

$$
\begin{array}{lll}
\{(x,y),(x+1,y)\} & \{(x,y),(x,y+1)\} & \{(x,y),(x-1,y+1)\} \\
\{(x,y),(x-1,y)\} & \{(x,y),(x,y-1)\} & \{(x,y),(x+1,y-1)\}
\end{array}
$$

This defines a maximal, closed umbrella–path.



If we consider the point $[(0, y)]$ with $y > \max(0, m - \lceil \frac{x}{2} \rceil)$, we obtain the following incident edges:

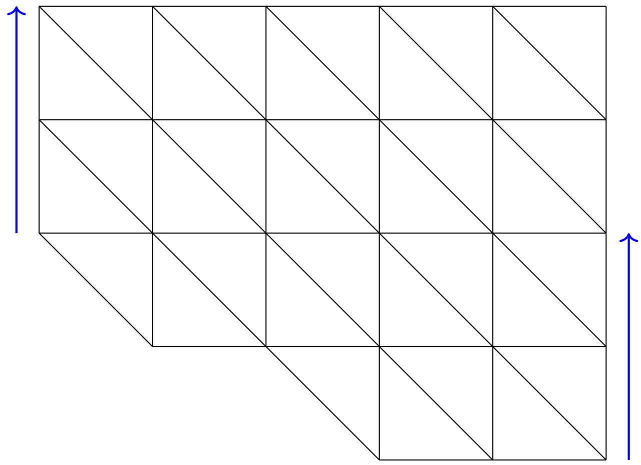
$$
\begin{array}{rr}
\{(0,y),(1,y-1)\} & \{(w,y-m),(w-1,y-m+1)\} \\
\{(0,y),(1,y)\} & \{(w,y-m),(w-1,y-m)\} \\
\{(0,y),(0,y+1)\} & \{(w,y-m),(w,y-m-1)\}
\end{array}
$$

This also defines a maximal, closed umbrella–path.

If $y = \max(0, m - \lceil \frac{x}{2} \rceil)$, the edges containing $y - 1$ are missing. Also, if $y + \frac{x}{2} = m$, the edge containing $(x - 1, y)$ is missing. In these cases, we obtain a boundary vertex. $\square$

Although Definition 8.3.6 seems daunting, the surface itself looks mostly harmless.

Example 8.3.7. *The hexagonal cylinder of waist–length 5 and offset 2 looks like this (infinitely extended upwards):*



If we consider the boundary of the hexagonal cylinder in Example 8.3.7, the degrees seem to form a staircase. The next remark shows that this always holds.

Remark 8.3.8. *Let C be a hexagonal cylinder with waist–length w and offset m. Let $\partial C = (V, E, \eta)$. The cyclic sequence*

$$V \to \mathbb{N} \qquad\qquad p \mapsto \deg(p) \tag{8.1}$$

is an m–staircase.

Proof. The boundary vertices of the cylinder are given by $(k, \max(0, m - \lceil \frac{k}{2} \rceil))$ (compare the well–definedness of Definition 8.3.6 for details).

We compute the degree of $(0, v)$. It is clearly incident to the face $\{(0, v), (1, v), (0, v + 1)\} \in F_+$. If we consider $(0, v) \sim (w, 0)$, we also see the incidence to the faces

$$\{(w, 0), (w, 1), (w - 1, 1)\} \in F_-$$
$$\{(w - 1, 0), (w, 0), (w - 1, 1)\} \in F_+,$$

where $(w - 1, 0) \in V$ since

$$m - \left\lceil \frac{w-1}{2} \right\rceil = \begin{cases} m - \frac{w}{2} \leq \frac{w}{2} - \frac{w}{2} = 0 & w \text{ even} \\ m - \frac{w-1}{2} \leq \frac{w-1}{2} - \frac{w-1}{2} & w \text{ odd} \end{cases}.$$

In the odd case, we obtain $m \leq \lfloor \frac{w}{2} \rfloor = \frac{w-1}{2}$.

If $m > 0$, the vertex $(0, m)$ is also incident to the face $\{(1, m-1), (0, m), (1, m)\} \in F_-$. Thus,

$$\deg([(0, m)]) = \begin{cases} 3 & m = 0 \\ 4 & m > 0. \end{cases}$$

In the case $m = 0$, every vertex $(x, 0)$ with $0 < x < w$ is incident to the three faces

$$\begin{aligned}
\{(x,0),(x+1,0),(x,1)\} &\in F_+\\
\{(x,0),(x,1),(x-1,1)\} &\in F_-\\
\{(x-1,0),(x,0),(x-1,1)\} &\in F_+,
\end{aligned}$$

which gives a 0–staircase.

If $m > 0$, the same argument is true for all $(x, 0)$, where $m - \frac{x}{2} \leq 0$. Consider the remaining vertices $(x, \lceil \frac{x}{2} \rceil)$. If x is odd, the vertex is incident to the two faces

$$\begin{aligned}
\left\{\left(x,\frac{x+1}{2}\right),\left(x+1,\frac{x+1}{2}\right),\left(x,1+\frac{x+1}{2}\right)\right\} &\in F_+\\
\left\{\left(x,\frac{x+1}{2}\right),\left(x,1+\frac{x+1}{2}\right),\left(x-1,1+\frac{x+1}{2}\right)\right\} &\in F_-.
\end{aligned}$$

If x is even, the vertex is incident to the four faces

$$\begin{aligned}
\left\{\left(x+1,\frac{x}{2}-1\right),\left(x+1,\frac{x}{2}\right),\left(x,\frac{x}{2}\right)\right\} &\in F_-\\
\left\{\left(x,\frac{x}{2}\right),\left(x+1,\frac{x}{2}\right),\left(x,1+\frac{x}{2}\right)\right\} &\in F_+\\
\left\{\left(x,\frac{x}{2}\right),\left(x,1+\frac{x}{2}\right),\left(x-1,1+\frac{x}{2}\right)\right\} &\in F_-\\
\left\{\left(x-1,\frac{x}{2}\right),\left(x,\frac{x}{2}\right),\left(x-1,\frac{x}{2}+1\right)\right\} &\in F_+.
\end{aligned}$$

Thus, we have an m–staircase. □

At this point, we can construct the infinite regular extension of extended SB–surfaces with boundary defect 0.

Theorem 8.3.9. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0. Then, $(S, \widehat{\deg})$ has an infinite regular extension.*

Proof. By Lemma 8.3.5, there is an extension $(U, \widehat{\deg}_U, (v_1, \ldots, v_k))$ of $(S, \widehat{\deg})$, such that $\widehat{\deg}_U$ is an m–staircase for some m. Let w be the number of vertices in the boundary graph ∂U.

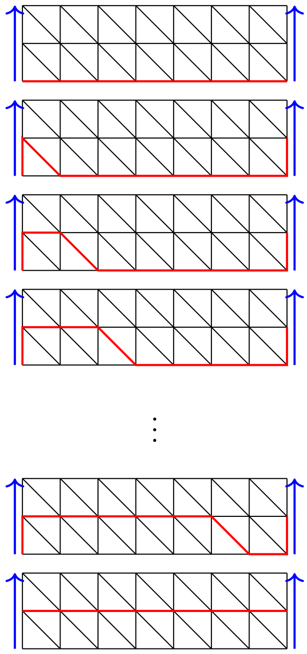
We combine (with Lemma 4.2.12) $(U, \widehat{\deg}_U)$ with a hexagonal cylinder. This is possible by Remark 8.3.8. To apply the uniqueness Theorem 8.2.11, we need to construct an infinite sequence of extensions. We distinguish three cases:

1. If $m = 0$, we add vertices in the following sequence:

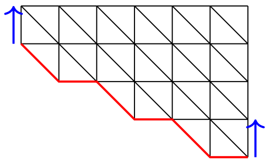
   $$(0,1),(1,1),(2,1),\ldots,(w-1,1),(0,2),(1,2),\ldots$$

   This leads to the following changes in the external degree sequence, all of which are growth–controlled:

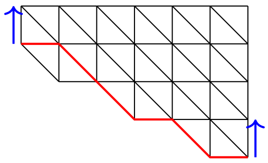
$$
\begin{gathered}
(3, \ldots, 3) \\
\downarrow \\
(5, 2, 3, \ldots, 3, 2) \\
\downarrow \\
(4, 4, 2, 3, \ldots, 3, 2) \\
\downarrow \\
(4, 3, 4, 2, 3, \ldots, 3, 2) \\
\downarrow \\
\vdots \\
\downarrow \\
(4, 3, \ldots, 3, 4, 2, 2) \\
\downarrow \\
(3, \ldots, 3)
\end{gathered}
$$



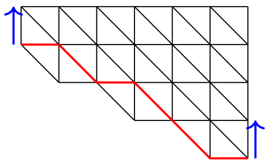
2. If $2m = w$, the external degree sequence has the form $(4, 2, 4, 2, 4, 2, \ldots, 4, 2)$.



Extending the second vertex gives $(3, 4, 3, 2, 4, 2, \ldots, 4, 2)$.

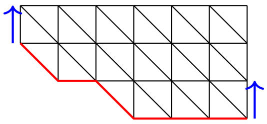


Extending the fourth vertex gives $(3, 4, 2, 4, 3, 2, \ldots, 4, 2)$.

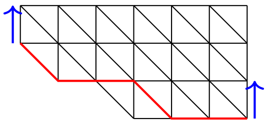


We continue until we have $(2, 4, 2, 4, \ldots, 2, 4)$. All these sequences are growth–controlled. We repeat the procedure for the vertices on first, third, fifth, etc. position.

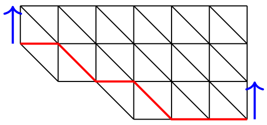
3. Let $0 < m < \frac{w}{2}$. By Remark 8.3.8 and Definition 8.3.4, there is an enumeration $u_1, u_2, \ldots, u_w$ of the boundary vertices of the hexagonal cylinder such that $u_2, u_4, \ldots, u_{2m}$ are the vertices with degree 2 (with respect to the cylinder).



By assumption, u_{2m+1} has degree 3. Thus, if we replace u_{2m} by $u_{2m} + (0, 1)$, the resulting external degree sequence remains a staircase.



We repeat this argument for u_{2m-2}, then w_{2m-4}, until u_2.



Shifting by one, $u_2, u_3, \dots, u_w, u_1$ is an m–staircase again. We repeat the argument from before. Since all vertices of a certain height are extended at some point, this is an appropriate infinite sequence. $\square$

The construction in Theorem 8.3.9 relies on Lemma 8.3.5, which constructs an object $(U, \widehat{\deg}_U)$ in the regular extension category, where $\widehat{\deg}_U$ is an m–staircase. The parameter w appears as the number of vertices in ∂U.

Both w and m appear to depend on the choice of $(U, \widehat{\deg}_U)$. The next lemma shows that they are independent from this choice.

Lemma 8.3.10. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0. If the infinite regular extension $\varprojlim S$ can be constructed with two different hexagonal cylinders, these cylinders have the same parameters w and m.*

Proof. Following the construction of the infinite regular extension in Theorem 8.3.9, there are vertex extensions $(T_1, \widehat{\deg}_{T_1})$ and $(T_2, \widehat{\deg}_{T_2})$ of $(S, \widehat{\deg})$, such that $T_1 +_{\rho_1} C_1 = \varprojlim S = T_2 +_{\rho_2} C_2$, for appropriate hexagonal cylinders C_1 and C_2, and graph isomorphisms ρ_1, ρ_2 like in Lemma 4.2.12.

We can shift the boundary of C_1 by $(0, k)$ for any $k \in \mathbb{N}$, until it gives a path in C_2. Thus, we only need to consider staircase–paths from $(0, n + m)$ to (w, n) (for any $n \in \mathbb{N}$). There are three kinds of edges: horizontal, vertical, and diagonal. For a path with out–degree 3, there are three options:

- Use two horizontal edges: This increases the length of the path by 1, and leaves the second component invariant.
- Use two vertical edges: This does not change the length, but changes the second component by 1.
- Use two diagonal edges: This increases both length and second component by 1.

Staircase–paths only use two of the three edge–types. If they use diagonal and vertical edges, then the offset would be equal to the waist–length, in contradiction to the fact that the offset can be at most half the waist–length.

If they use horizontal and diagonal edges, we obtain $k = m$.

If they use horizontal and vertical edges, this gives a staircase in the opposite direction, which never closes. □

Lemma 8.3.10 motivates us to define waist–length and offset as properties of the extended SB–surface. Comparing these parameters to (l, m) from the classification of nanotubes in [17] tells us that the waist–length is $l + m$ and the offset is equal to m.

Definition 8.3.11. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0, whose infinite regular extension is built with a hexagonal cylinder of waist–length w and offset m.*

*The **waist–length** of $(S, \widehat{\deg})$ is w and the **offset** of $(S, \widehat{\deg})$ is m.*

Up to this point, the construction of the infinite regular extension has been done very concretely. Now, we broaden our horizon a bit: We allow the combination of an extended SB–surface $(S, \widehat{\deg})$, where $\widehat{\deg}$ is a staircase, with a subsurface of the hexagonal cylinder.

Lemma 8.3.12. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0. Let $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$ be an object of the regular extension category such that $\widehat{\deg}_T$ is a staircase. Then, $\varprojlim S$ is the combination of T and a subsurface of a hexagonal cylinder, in the sense of Lemma 4.2.12.*

Proof. By Lemma 8.3.5, there is an object $(U, \widehat{\deg}_U, (v_1, \ldots, v_n, w_1, \ldots, w_k))$ of the regular extension category such that $\widehat{\deg}_U$ is an m–staircase. Similar as in Theorem 8.3.9, we combine U with the subsurface C of the hexagonal cylinder (V, E, F, η, φ) induced by the vertices

$$\{(x, y) \in V \mid y \geq k + y_0, \text{ with } y_0 := \min\{\hat{y} \mid (x, \hat{y}) \in V\}\}.$$

We can interpret w_k as boundary vertex of both U and C. By construction, w_k is a regular vertex. If we remove w_k from U (forming U^{-w_k}) and simultaneously extend C along this vertex (forming C^{+w_k} such that w_k remains regular), the combined surface is unchanged. However, it is now represented as the combination of $(U', \widehat{\deg}_{U'}, (v_1, \ldots, v_n, w_1, \ldots, w_{k-1}))$ and a subsurface of the hexagonal cylinder.

We repeat this argument until $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$. By definition of the subsurface C, the constructed extension is still a subsurface of the hexagonal cylinder. □

With this more general conception, we can ask for the "smallest" possible extension that allows this construction.

Definition 8.3.13. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0. An element $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$ of the regular extension category is called **tight** if*

1. $\widehat{\deg}_T$ *is a staircase.*

2. *There is no other* $(U, \widehat{\deg}_U, (w_1, \ldots, w_k))$*, where* $\widehat{\deg}_U$ *is a staircase, such that there is a morphism from* $(U, \widehat{\deg}_U, (w_1, \ldots, w_k))$ *to* $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$*.*

An element $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$ *of the regular extension category is called* v*–**tight** if*

1. $\widehat{\deg}_T$ *is a* v*–staircase.*

2. *There is no other* $(U, \widehat{\deg}_U, (w_1, \ldots, w_k))$*, where* $\widehat{\deg}_U$ *is a* v*–staircase, such that there is a morphism from* $(U, \widehat{\deg}_U, (w_1, \ldots, w_k))$ *to* $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$*.*

In Definition 8.3.13, the minimality is defined via the partial ordering of the regular extension category. But actually, the concept of tightness denotes a global minimum.

Lemma 8.3.14. *Let* $(S, \widehat{\deg})$ *be a growth–controlled extended SB–surface with boundary defect 0. Then, there is a unique tight element in the regular extension category of* $(S, \widehat{\deg})$*.*

Proof. Since there are elements of the regular extension category whose external degree sequences are staircases, and the partial order implicitly defined in Definition 8.3.13 is bounded from below, there exists a tight element.

Consider such an element. By Lemma 8.3.12, we can associate a path P in the hexagonal cylinder with it. Any other tight element has a path that crosses P in at least one vertex (otherwise one would be strictly smaller).

Since the paths cannot use the vertical edges (then they would not be staircases), there are only two options for path behaviour:

- Two horizontal edges.
- A horizontal edge and a diagonal edge.

If the paths cross, they have to have one horizontal edge in common. Consider this edge and look to the first separation to the left: One path takes a horizontal edge and the other a diagonal one. But then the path taking the diagonal edge could be tightened (since the vertex below has degree 6).

Since the path is minimal, this cannot happen. The only possibility for this is that the out–degree–sequence has the subsequence $(4, 2, 4, 2)$ at this point. But in this case, the lower path has to stay strictly below the upper one, as long as the the upper path continues with the sequence of $(4, 2)$. Therefore, all of those could be shifted down one step, which is a contradiction. □

While tight elements are unique, this is not true for v–tight elements. However, they do possess some structure. To explore this, we need the concept of *area*.

Definition 8.3.15. *Let* $(S, \widehat{\deg})$ *be an extended SB–surface and* $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$ *an object of its regular extension category. The number of faces in* T *is the **area** of this object.*

We can use the concept of *area* to describe the difference between two m–tight elements.

Lemma 8.3.16. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0, waist–length w and offset m. Let E_1 and E_2 be two m–tight elements in the regular extension category. Then, the difference between their areas is divisible by $2\gcd(w, m)$.*

Proof. For $m = 0$, there is nothing to show since there is only one 0–tight element.

Otherwise, by Lemma 8.3.14, there is a unique tight element $(T, \widehat{\deg}_T, (v_1, \dots, v_n))$ with unique morphisms to E_1 and E_2. This allows us to describe the boundaries of E_1 and E_2 as closed paths (call them P_1 and P_2) in a subsurface of the hexagonal cylinder (since by Lemma 8.3.12), the complement of T in $\varprojlim S$ can be described by such a subsurface).

Then, the external degree sequence of $(T, \widehat{\deg}_T)$ has the form

$$(2, 4, 3^{e_1}, 2, 4, 3^{e_2}, \dots, 2, 4, 3^{e_{v-1}}, 2, 4, 3^{e_v}),$$

where we interpret 3^k as k copies of 3.

Let P be the path associated to the boundary of T. Then, P_1 and P_2 have to intersect P in at least one $(2, 4)$–pair (otherwise they are not tight). We can construct an m–tight path in the following way:

1. Extend along the second $(2, 4)$–pair to obtain the external degree sequence

   $$(2, 4, 3^{e_1 - 1}, 2, 4, 3^{e_2 + 1}, \dots, 2, 4, 3^{e_{v-1}}, 2, 4, 3^{e_v}),$$

   whose area is increased by 2. Repeat this until we have

   $$(2, 4, 2, 4, 3^{e_1 + e_2}, \dots, 2, 4, 3^{e_{v-1}}, 2, 4, 3^{e_v}),$$

   which increases the area by $2e_1$ in total.

2. Repeat the process for the third $(2, 4)$–pair, to obtain

   $$(2, 4, 2, 4, 2, 4, 3^{e_1 + e_2 + e_3}, \dots, 2, 4, 3^{e_{v-1}}, 2, 4, 3^{e_v}),$$

   with total increased are $2e_1 + 2(e_1 + e_2)$.

3. Repeat the process for all remaining pairs.

Of course, we could start this process at any pair in the sequence. Any m–tight path arises in one of these fashions. If we start at e_j (and interpret the indices modulo m), the total area increase is

$$2 \sum_{i=1}^{m-1} \sum_{k=1}^{i} e_{j+k} = 2 \sum_{i=1}^{m-1} (m - i) e_{j+i} = 2 \sum_{i=1}^{m} (m - i) e_{j+i}. \tag{8.2}$$

Now we compute the difference between two of these sums (corresponding to the area difference between two m–tight paths). Our starting positions are j and $j+t$ with $1 \le t < m$. This gives

$$2\left(\sum_{i=1}^{m}(m-i)e_{j+i} - \sum_{i=1}^{m}(m-i)e_{j+t+i}\right) = 2\left(\sum_{i=1}^{m}(m-i)e_{j+i} - \sum_{i=t+1}^{m+t}(m+t-i)e_{j+i}\right)$$

Since $1 \le t < m$ we have $2 \le t+1 \le m$, so we can split the sums:

$$\begin{aligned}
&= 2\left(\sum_{i=1}^{t}(m-i)e_{j+i} + \sum_{i=t+1}^{m}(m-i)e_{j+i} - \sum_{i=t+1}^{m}(m+t-i)e_{j+i} - \sum_{i=m+1}^{m+t}(m+t-i)e_{j+i}\right)\\
&= 2\left(\sum_{i=1}^{t}(m-i)e_{j+i} - t\sum_{i=t+1}^{m}e_{j+i} - \sum_{i=1}^{t}(t-i)e_{j+i+m}\right)\\
&= 2\left(\sum_{i=1}^{t}(m-i-t+i)e_{j+i} - t\sum_{i=t+1}^{m}e_{j+i}\right)\\
&= 2\left((m-t)\sum_{i=1}^{t}e_{j+i} - t\sum_{i=t+1}^{m}e_{j+i}\right),
\end{aligned}$$

where we have used that $e_{j+i+m} = e_{j+i}$.

From our description of T we deduce

$$2m + \sum_{i=1}^{m} e_{j+i} = w \qquad \Leftrightarrow \qquad \sum_{i=t+1}^{m} e_{j+i} = w - 2m - \sum_{i=1}^{t} e_{j+i}.$$

We use this to further simplify our difference.

$$\begin{aligned}
&= 2\left((m-t)\sum_{i=1}^{t}e_{j+i} - t\left(w-2m-\sum_{i=1}^{t}e_{j+i}\right)\right)\\
&= 2\left(m\sum_{i=1}^{t}e_{j+i} - t(w-2m)\right)
\end{aligned}$$

This difference is always divisible by $2\gcd(m, w-2m) = 2\gcd(m,w)$. $\square$

Lemma 8.3.16 tells us that the area of a m–tight element is an invariant, if we only consider it modulo $2\gcd(m,w)$. Of course, this is sometimes not helpful at all (for $\gcd(m,w)=1$). But in many cases, it subdivides SB–surfaces of the same waist–length and offset quite nicely.

Definition 8.3.17. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect 0, waist–length w and offset m. The* ***staircase area*** *of S is the area of any m–tight object in the regular extension category, interpreted as a number in $\mathbb{Z}/2\gcd(w,m)\mathbb{Z}$.*

The staircase area is important since it changes predictably with local modifications (we only have to count the difference between the faces before and after the modification).

For concreteness, we look at the vertex split from Section 8.1, again.

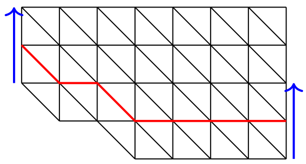
Corollary 8.3.18. *Let $(S, \widehat{\deg}_S)$ and $(T, \widehat{\deg}_T)$ be two growth–controlled extended SB–surfaces with boundary defect 0, such that $(T, \widehat{\deg}_T)$ is constructed from splitting a vertex of $(S, \widehat{\deg}_S)$. Then, the staircase area of T is the staircase area of S plus 2.*

Proof. Let w be the waist–length and m be the offset.

Pick an m–tight object from the regular extension category of $(S, \widehat{\deg}_S)$. Performing the vertex splits changes it into an extended SB–surface $(U, \widehat{\deg}_U)$. This extended SB–surface is not necessarily m–tight in the regular extension category of $(T, \widehat{\deg}_T)$. If it is not, we construct a “smaller” m–tight object. Let A be the staircase area of $(S, \widehat{\deg}_S)$.

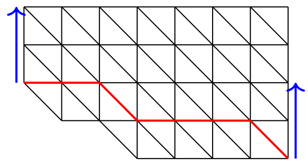
- If $m = 0$, we can remove all boundary vertices to obtain an object whose external degree sequence is a 0–staircase. Its area is $A + 2 - 2w$.
- If $m > 0$, we enumerate the vertices of ∂U as $v_1, v_2, \ldots, v_w$ like in Definition 8.3.4. There are two possibilities to construct a “smaller” m–tight object:
  1. If none of $v_1, v_3, \ldots, v_{2m-1}$ (the vertices with external degree 4) are singular, they can all be reduced to produce an m–tight object with area $A + 2 - 2m$.
  2. If v_1 is regular, removing v_1 changes the external degree sequence from

     $$(4, 2, 4, 2, \ldots, 4, 2, 3, \ldots, 3)$$

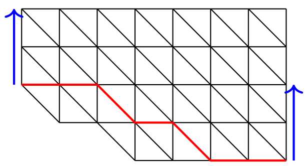
     to

     $$(2, 3, 4, 2, \ldots, 4, 2, 3, \ldots, 3, 4)$$

     If we can repeat this process $w - 2m$ times, we obtain another m–tight object with external degree sequence

     $$(3, 3, 4, 2, \ldots, 4, 2, 3, \ldots, 3)$$

     Its area is $A + 2 - 2w + 4m$.

Since every element in $\{2w, 2m, 2w - 4m\}$ is divisible by $2\gcd(w, m)$, the staircase area of $(T, \widehat{\deg}_T)$ is $A + 2$. □

Thus, the subdivision defined by the staircase area carries a natural cyclic grading with respect to vertex splitting.

8.3.2 Negative boundary defect

In this subsection, we perform the explicit construction of the infinite regular extension for extended SB–surfaces (recall Definition 3.4.17) with negative boundary defect (Definition 8.3.2). Roughly, the extension is constructed from the SB–surface and several cones. These extensions correspond to *nanocones* which are classified in [18]. Brinkmann and Van Cleemput used geometric arguments to achieve this classification. We develop a method relying on a group action (the *growth action*).

Similar to Subsection 8.3.1, we start by constructing a vertex–extension with "nice" external degree sequence.

Lemma 8.3.19. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with negative boundary defect. There exists an element $(T, \widehat{\deg}_T, (w_1, \dots, w_k))$ in the regular extension category such that its external degree sequence only takes the values 3 and 4. In addition, the value 4 is taken as often as the absolute value of the boundary defect.*

Proof. Applying Lemma 8.3.3 gives a growth–controlled extended SB–surface $(U, \widehat{\deg}_U)$, where the defect–sum of every cyclic interval is at most 1. Consider three adjacent vertices (v_1, v_2, v_3) in the boundary ∂U such that $\widehat{\deg}_U(v_1) = 5$.

- If $\widehat{\deg}_U(v_2) = 2$, we know $\widehat{\deg}_U(v_3) > 2$ (otherwise the cyclic interval induced by $\{v_2, v_3\}$ would have defect–sum 2). Thus, the extension of Lemma 6.2.8 is possible and growth–controlled by Lemma 6.2.9. The extended degree subsequence $(5, 2, \widehat{\deg}_U(v_3))$ is replaced by $(4, 4, \widehat{\deg}_U(v_3) - 1)$.
- If $\widehat{\deg}_U(v_2) > 2$, the extension of Lemma 6.2.3 is applicable and growth–controlled by Lemma 6.2.5. It changes the degree subsequence from $(5, \widehat{\deg}_U(v_2), \widehat{\deg}_U(v_3))$ to $(4, 5, \widehat{\deg}_U(v_2) - 1, \widehat{\deg}_U(v_3))$.

  We repeat this argument for the subsequence $(5, \widehat{\deg}_U(v_2), \widehat{\deg}_U(v_3))$ until the first case is applicable.

In any case, the number of vertices with value 5 under $\widehat{\deg}_U$ is reduced by 1. We apply Lemma 8.3.3 again to regain the property that the defect–sum of cyclic intervals is at most 1. Since this extension does not increase the number of vertices with value 5, we can repeat this process until all vertices have values 2, 3, or 4.

We remove the vertices with value 2 iteratively. Let $(v_1, \dots, v_j, \dots, v_n)$ be a sequence of boundary vertices such that

$$\widehat{\deg}_U(v_1) = \widehat{\deg}_U(v_n) = 4, \qquad \widehat{\deg}_U(v_j) = 2, \qquad \widehat{\deg}_U(v_k) = 3 \text{ otherwise.}$$

We show that there is a vertex extension where the number of vertices with value 2 is reduced by one. To follow this process, we describe the external degree sequence as tuple $(4, 3, \dots, 3, 2, 3, \dots, 3, 4)$. We proceed in three steps:

1. If $\widehat{\deg}_U(v_2) = \widehat{\deg}_U(v_{n-1}) = 3$, we use Lemma 6.2.8 to extend the surface. This modifies the external degree sequence into

$$(4, 3, \dots, 3, 2, 4, 2, 3, \dots, 3, 4).$$

By repeating this extension for both vertices with external degree 2, we obtain the external degree sequence

$$(4, 3, \dots, 3, 2, 4, 2, 4, 2, 3, \dots, 3, 4).$$

We apply the extension to all vertices with external degree 2 repeatedly, until we arrive at one of the following sequences:

$$(4, 2, 4, 2, \dots, 2, 4) \qquad\qquad (4, 3, \dots, 3, 2, 4, 2, 4, \dots, 2, 4)$$

2. Assume that the external degree sequence has the form

$$(4, 3, \dots, 3, 2, 4, 2, 4, \dots, 2, 4).$$

If we extend the surface with Lemma 6.2.8 at each vertex with value 2, we obtain the sequence

$$(4, 3, \dots, 3, 2, 4, 2, 4, \dots, 2, 4, 3).$$

By repeating this argument, we arrive at the sequence $(4, 2, 4, \dots, 4, 2, 4)$.

3. Assume that the external degree sequence has the form

$$(4, 2, 4, \dots, 4, 2, 4).$$

If we extend the surface by Lemma 6.2.8 at each vertex of value 2, we obtain the sequence

$$(3, 4, 2, \dots, 2, 4, 3).$$

We repeat this process until we stop at the sequence $(3, \dots, 3, 4, 3, \dots, 3)$.

Repeating this process for all vertices with value 2 results in an extended SB–surface whose external degree sequence only takes the values 3 and 4.

The additional claim about the number of 4s taken is a direct consequence of Definition 8.3.2 of the boundary defect. □

Lemma 8.3.19 shows how to construct a "nice" vertex–extension. Next, we have to define an infinite simplicial surface. In contrast to Subsection 8.3.1, the surface is more complicated for negative boundary defect. It consists of several "cones" that are glued together. Thus, we start by defining these "cones" or *slices*. Figures 8.2 and 8.3 show illustrations of the slices.

Definition 8.3.20. *The* ***hexagonal slice with base length*** b *is the vertex–faithful simplicial surface* (V, E, F, η, φ) *with:*

- $V = \{(x, y) \in \mathbb{Z}^2 \mid b \leq x, 0 \leq y \leq x\}$.
- $E = E_{-} \uplus E_{/} \uplus E_{|}$ *with*

$$\begin{aligned} E_{-} &= \{\{(x, y), (x + 1, y)\} \mid (x, y) \in V\} \\ E_{/} &= \{\{(x, y), (x + 1, y + 1)\} \mid (x, y) \in V\} \\ E_{|} &= \{\{(x, y), (x, y + 1)\} \mid (x, y) \in V, y < x\}. \end{aligned}$$

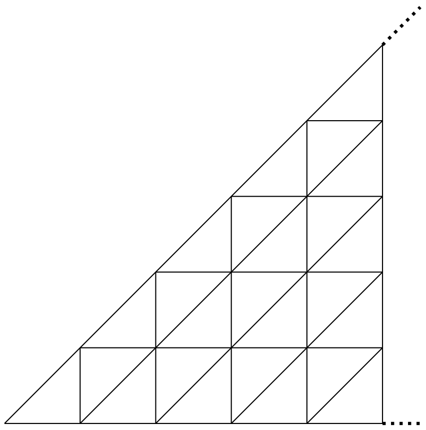


Figure 8.2: hexagonal slice with $b = 0$

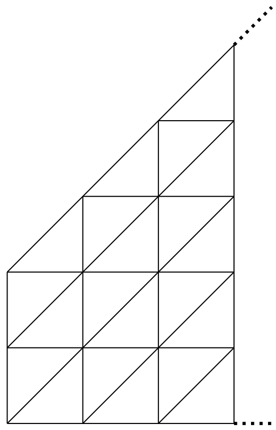


Figure 8.3: hexagonal slice with $b = 2$

- $F = F_+ \uplus F_-$ *with*

$$\begin{aligned} F_+ &= \{\{(x,y),(x+1,y),(x+1,y+1)\} \mid (x,y) \in V\} \\ F_- &= \{\{(x,y),(x,y+1),(x+1,y+1)\} \mid (x,y) \in V, y < x\}. \end{aligned}$$

- $\eta : E \to \mathrm{Pot}_2(V)$ *is defined by* $\eta(e) = e$.
- $\varphi : F \to \mathrm{Pot}_3(E)$ *is defined by* $\varphi(f) = \{e \in E \mid e \subseteq f\}$.

Well–defined. We have to show that the hexagonal slice is a vertex–faithful simplicial surface. To do so, we check the prerequisites of Lemma 2.7.5.

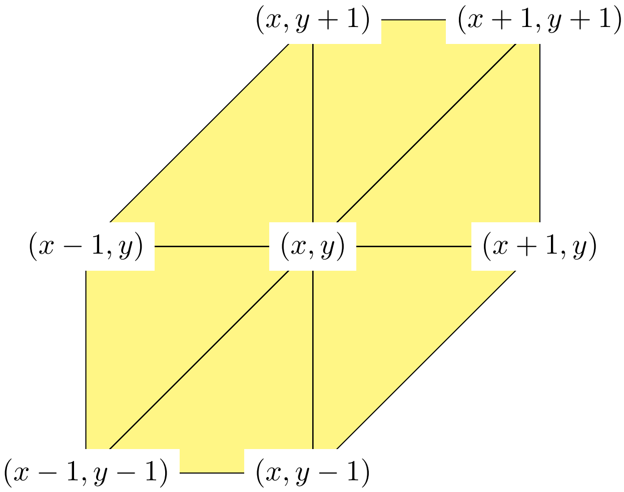
1. Clearly, η and $\eta \taurus \varphi$ are injective.
2. The vertex (x,y) lies in the edge $\{(x,y),(x+1,y)\}$.
3. The edges $\{(x,y),(x+1,y)\}$ and $\{(x,y),(x+1,y+1)\}$ lie in the face $\{(x,y),(x+1,y),(x+1,y+1)\}$. The edge $\{(x,y),(x,y+1)\}$ lies in the face $\{(x,y),(x,y+1),(x+1,x+1)\}$.
4. By construction, every face has three edges.

Thus, the hexagonal slice is a vertex–faithful triangular complex.

Next, we have to show that there are neither edge ramifications nor vertex ramifications. Clearly, each edge can lie in at most two faces, so there are no ramified edges.

To show that there are no ramified vertices, consider the vertex (x,y). In general, such a vertex is incident to six edges and six faces, which we can describe by the adjacent edges:

$$\begin{array}{lll} \{(x,y),(x+1,y)\} \in E_- & \{(x,y),(x+1,y+1)\} \in E_/ & \{(x,y),(x,y+1)\} \in E_| \\ \{(x,y),(x-1,y)\} \in E_- & \{(x,y),(x-1,y-1)\} \in E_/ & \{(x,y),(x,y-1)\} \in E_|. \end{array}$$



Thus, a vertex (x, y) with $x > b$ and $y \notin \{0, x\}$ is an inner vertex. In the corner cases, we obtain fewer faces:

- If $x = b$, the two edges to $(x-1, y)$ and $(x-1, y-1)$ are missing.
- If $y = 0$, the two edges to $(x-1, y-1)$ and $(x, y-1)$ are missing.
- If $y = x$, the two edges to $(x, y+1)$ and $(x-1, y)$ are missing.

It is important to note that all combinations of these restrictions lead to a unique maximal umbrella. Thus, (V, E, F, η, φ) is a simplicial surface. $\square$

Since we want to combine several of these slices into a larger surface, it is helpful to know the umbrella paths of each vertex explicitly. Since these have already been described in the well–definedness of Definition 8.3.20, we just restate those results:

Remark 8.3.21. *Let (V, E, F, η, φ) be a hexagonal slice with base length b. Let $(x, y) \in V$. The umbrella–path of (x, y) can be described by a sequence of adjacent vertices $(v_1, \ldots, v_k)$:*

- *The edges of the path are $e_i := \{(x, y), v_i\}$ for $1 \le i \le k$.*
- *The faces of the path are $f_i := \{(x, y), v_i, v_{i+1}\}$ for $1 \le i < k$. If the path is closed, $f_k := \{(x, y), v_k, v_1\}$ also is a face.*

With this convention, we have:

- *$(x, y) = (0, 0)$ (only for $b = 0$) is a boundary vertex, whose umbrella–path is described by*
$$((1, 0), (1, 1)).$$
- *$(x, y) = (b, 0)$ with $b > 0$ is a boundary vertex, whose umbrella–path is described by*
$$((b+1, 0), (b+1, 1), (b, 1)).$$

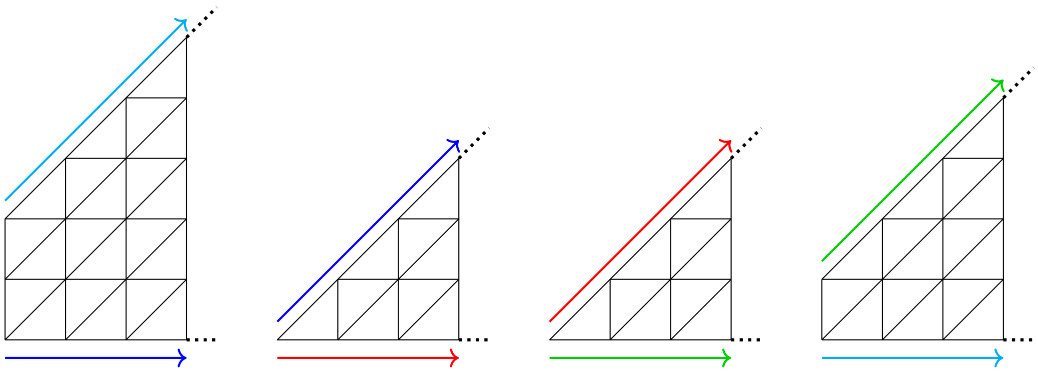


Figure 8.4: Hexagonal cake with base lengths $(2, 0, 0, 1)$

- $(x, y) = (b, b)$ *with* $b > 0$ *is a boundary vertex, whose umbrella–path is described by*

$$((b, b-1), (b+1, b), (b+1, b+1)).$$

- $(x, y) = (x, 0)$ *with* $x > b$ *is a boundary vertex, whose umbrella–path is described by*

$$((x+1, 0), (x+1, 1), (x, 1), (x-1, 0)).$$

- $(x, y) = (x, x)$ *with* $x > b$ *is a boundary vertex, whose umbrella–path is described by*

$$((x-1, x-1), (x, x-1), (x+1, x), (x+1, x+1)).$$

- (x, y) *satisfying none of the cases above is an inner vertex, whose umbrella–path is described by*

$$((x-1, y-1), (x, y-1), (x+1, y), (x+1, y+1), (x, y+1), (x-1, y)).$$

We can combine several hexagonal slices to construct an infinite extension. Since this combination has a cyclic nature, it is reminiscent of a *cake*, which motivates the name of the surface. One of these surfaces is illustrated in Figure 8.4.

Definition 8.3.22. *Let* $(b_1, b_2, \dots, b_n) \in (\mathbb{Z}_{\geq 0})^n$ *(with* $n \geq 1$*) and* $S_i = (V_i, E_i, F_i, \eta_i, \varphi_i)$ *hexagonal slices with base length* b_i*, such that* $\sum_{i=1}^n b_i \geq 2$*. The* ***hexagonal cake with base lengths*** $(b_1, \dots, b_n)$ *is the simplicial surface* $(\biguplus V_i / \sim_V, \biguplus E_i / \sim_E, \biguplus F_i, \eta, \varphi)$ *with*

- $\sim_V$ *is an equivalence relation on* $\biguplus V_i$*, such that* $(k + b_i, 0) \in V_i$ *is equivalent to* $(k + b_{i+1}, k + b_{i+1}) \in V_{i+1}$*, where we read indices modulo* n*.*
- $\sim_E$ *is an equivalence relation on* $\biguplus E_i$*, such that* $\{(k + b_i, 0), (k + 1 + b_i, 0)\} \in E_i$ *is equivalent to* $\{(k + b_{i+1}, k + b_{i+1}), (k + 1 + b_{i+1}, k + 1 + b_{i+1})\} \in E_{i+1}$*, where we read indices modulo* n*.*

- *η maps the edge $[e]_{\sim_E}$ with $e \in E_i$ to $\{[v]_{\sim_V} \mid v \in \eta_i(e)\}$.*
- *φ maps the face $f \in F_i$ to $\{[e]_{\sim_E} \mid e \in \varphi_i(f)\}$.*

Well–defined. We have to show that the hexagonal cake is a well–defined simplicial surface.

We start by remarking that η is well–defined by definition of $\sim_V$ and $\sim_E$. Next, we check the conditions for polygonal complexes from Definition 2.5.2.

1. Since each face lies in one of the slices S_i, there is an alternating sequence of incident vertices and edges, $(v_1, e_1, v_2, e_2, v_3, e_3)$, with $\{v_1, v_2, v_3\} \subseteq V_i$ and $\{e_1, e_2, e_3\} \subseteq E_i$. It remains to show that $\{[v_1]_{\sim_V}, [v_2]_{\sim_V}, [v_3]_{\sim_V}\}$ and $\{[e_1]_{\sim_E}, [e_2]_{\sim_E}, [e_3]_{\sim_E}\}$ consist of three elements each.

   By definition of $\sim_E$, it suffices to show that two vertices that are incident to the same face in S_i are not equivalent under $\sim_V$.

   Consider $\{(x,y),(x+1,y),(x+1,y+1)\} \in F_+$. Since the difference between the first component and the base length of the slice is invariant under $\sim_V$ (denoted *height* in Definition 8.3.29), we only have to prove that $(x+1,y) \sim_V (x+1,y+1)$ is impossible. The only possible case where these two vertices are identified with other vertices is $(x,y) = (0,0)$, which implies $b_i = 0$.

   Since the vertices are not identified directly, each of the S_j (for $j \neq i$) has to have a vertex that is identified with vertices in S_{j-1} and S_{j+1}. The only such vertex is $(0,0)$. If all S_j have this vertex, we have $b_j = 0$, which implies $b_1 + \cdots + b_n = 0$, in contradiction to our assumption.

   Now we consider the other possible face $\{(x,y),(x,y+1),(x+1,y+1)\} \in F_-$. We obtain the equivalence $(x,y) \sim_V (x,y+1)$, which leads to $(x,y) = (1,0)$ and $b_i \leq 1$. By the same argument as above, $b_j = 0$ for $j \neq i$, so $\sum_{k=1}^{n} b_k \leq 1$, which is impossible.

2. Clearly, every vertex lies in an edge, and every edge lies in a face.

Since all faces in the S_i are triangular, this holds for the hexagonal cake as well.

We have to show that the hexagonal cake has no ramified edges. This only needs to be proven for those edges that lie in a non–trivial $\sim_E$–class. Consider the edge $\{(k+b_i,0),(k+1+b_i,0)\} \in E_i$. It is incident to a unique face of S_i and $\sim_E$–equivalent to the edge $\{(k+b_{i+1},k+b_{i+1}),(k+1+b_{i+1},k+1+b_{i+1})\} \in E_{i+1}$, which is incident to exactly one face of S_{i+1}. Thus, all of these edges are inner edges.

Finally, we have to show that the hexagonal cake has no ramified vertices. This only needs to be proven for those vertices that lie in a non–trivial $\sim_V$–class.

First, we consider the case in which the $\sim_V$–class contains exactly two elements, say $(k+b_i,0) \in V_i$ and $(k+b_{i+1},k+b_{i+1}) \in V_{i+1}$.

- If $k = 0$, we have $b_i \neq 0 \neq b_{i+1}$ (otherwise there would be more than two elements in the $\sim_V$–class). Remark 8.3.21 describes the umbrellas of these boundary vertices as

$$
\begin{aligned}
((b_i+1,0),(b_i+1,1),(b_i,1)) &\in V_i^3 \\
((b_{i+1},b_{i+1}-1),(b_{i+1}+1,b_{i+1}),(b_{i+1}+1,b_{i+1}+1)) &\in V_{i+1}^3.
\end{aligned}
$$

Since $(b_i+1,0) \sim_V (b_{i+1}+1,b_{i+1}+1)$, these umbrellas form a boundary vertex in the hexagonal cake.

- If $k > 0$, Remark 8.3.21 describes the umbrella of $(k+b_i,0)$ as

$$
\begin{gathered}
((k+b_i+1,0),(k+b_i+1,1), \\
(k+b_i,1),(k+b_i-1,0))
\end{gathered}
$$

and the umbrella of $(k+b_{i+1},k+b_{i+1})$ as

$$
\begin{gathered}
((k+b_{i+1}-1,k+b_{i+1}-1),(k+b_{i+1},k+b_{i+1}-1), \\
(k+b_{i+1}+1,k+b_{i+1}),(k+b_{i+1}+1,k+b_{i+1}+1)).
\end{gathered}
$$

Since $(k+b_i+1,0) \sim_V (k+b_{i+1}+1,k+b_{i+1}+1)$ and $(k+b_i-1,0) \sim_V (k+b_{i+1}-1,k+b_{i+1}-1)$, this forms an inner vertex.

Next, we consider the case where the $\sim_V$–class contains more than two elements. In this case, we have $(b_i,0) \in V_i$, identified with $(0,0) \in V_k$ for $i < k < j$ and $(b_j,b_j) \in V_j$ for some $j > i$ (where we read indices modulo n). Then, Remark 8.3.21 gives the umbrellas

$$
\begin{aligned}
((b_i+1,0),(b_i+1,1),(b_i,1)) &\in V_i^3 \\
((1,0),(1,1)) &\in V_k^2 \\
((b_j,b_j-1),(b_j+1,b_j),(b_j+1,b_j+1)) &\in V_j^3
\end{aligned}
$$

These umbrellas are combined via $\sim_V$ to form a boundary vertex of the cake. □

To combine a hexagonal cake with the vertex–extension of an extended SB–surface $(S,\widehat{\deg})$, we need to know the boundary of the cake.

Lemma 8.3.23. *Let (V,E,F,η,φ) be the hexagonal cake with base lengths $(b_1,\ldots,b_n) \in \mathbb{N}^n$. It has exactly one connected boundary component with degree–sequence*

$$
(4,3^{b_1-1},4,3^{b_2-1},\ldots,4,3^{b_n-1}),
$$

where 3^k represents k copies of 3.

Proof. From the well–definedness of Definition 8.3.22, we obtain that the boundary vertices of the hexagonal cake come from the vertices $(b_i,y) \in V_i$, where $(V_i,E_i,F_i,\eta_i,\varphi_i)$ are the hexagonal slices from Definition 8.3.22.

Since $b_i > 0$ for all $1 \leq i \leq n$, all $\sim_V$–classes contain at most two elements. A boundary vertex containing exactly one element has degree 3 (same degree as in the hexagonal slice). There are $b_i - 1$ of that in the hexagonal slice with base b_i. The vertices with exactly two elements have degree 4. □

With this, we can prove that infinite extensions exist for negative boundary defects.

Theorem 8.3.24. *Let* $(S, \widehat{\deg})$ *be a growth–controlled disc extended SB–surface with negative boundary defect. Then,* $(S, \widehat{\deg})$ *has an infinite regular extension.*

Proof. By Lemma 8.3.19, there is an element $(U, \widehat{\deg}_U, (v_1, \dots, v_k))$ in the regular extension category of $(S, \widehat{\deg})$, such that $\widehat{\deg}_U$ only takes the values 3 and 4 on ∂U.

We combine (with Lemma 4.2.12) $(U, \widehat{\deg}_U)$ with a hexagonal cake that has appropriate parameters. This is possible by Lemma 8.3.23.

To finish the proof, we apply the uniqueness Theorem 8.2.11 and need to construct an infinite sequence of vertex–extensions. Assume the external degree sequence of $(U, \widehat{\deg}_U)$ has the form

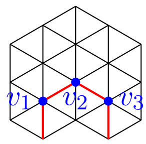
$$(4, 3^{b_1}, 4, 3^{b_2}, 4, 3^{b_3}, \dots, 4, 3^{b_n}),$$

where 3^k stands for a sequence of k 3s.

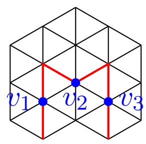
If $b_i = 0$ for all $1 \leq i \leq n$, the external degree sequence has the form $(4, 4, \dots, 4)$. It has at least length 3:

- Length 0 is impossible since the boundary defect is negative.
- Length 1 is impossible since there would have to be an edge with only one incident vertex, in contradiction to Definition 2.5.2.
- Length 2 is impossible since there would be two edges with the same incident vertices, which is a contradiction since U is vertex–faithful.

Consider any boundary subsequence (v_1, v_2, v_3).



We apply Lemma 6.2.3 to extend the edges between the vertex–pairs $\{v_1, v_2\}$ and $\{v_2, v_3\}$. These extensions are growth–controlled by Lemma 6.2.5 and they change the external degree sequence from $(\dots, 4, 4, 4, \dots)$ to $(\dots, 3, 5, 2, 5, 3, \dots)$.

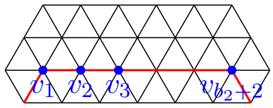


A final extension at the vertex v_2 (according to Lemma 6.2.8) gives the external degree sequence $(\dots, 3, 4, 4, 4, 3, \dots)$, which has the form

$$(4, 3^{b_1}, 4, 3^{b_2}, 4, 3^{b_3}, \dots, 4, 3^{b_n}).$$

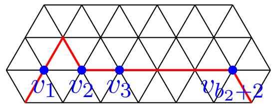
In contrast to before, now $b_i \geq 1$ for at least one $1 \leq i \leq n$.

Without loss of generality, $b_2 \geq 1$. Let $(v_1, \dots, v_{b_2+2})$ be the vertices corresponding to the subsequence $(4, 3^{b_2}, 4)$.



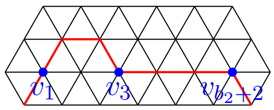
We extend the edge between v_1 and v_2 according to Lemma 6.2.3, with the external degree sequence

$$(4, 3^{b_1}, 3, 5, 2, 3^{b_2-1}, 4, 3^{b_3}, \dots, 4, 3^{b_n})$$



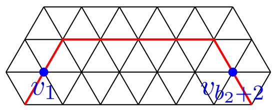
Next, we extend along v_2, according to Lemma 6.2.8 to obtain the external degree sequence

$$(4, 3^{b_1}, 3, 4, 4, 2, 3^{b_2-2}, 4, 3^{b_3}, \dots, 4, 3^{b_n})$$



We continue in this vein until we arrive at

$$(4, 3^{b_1+1}, 4, 3^{b_2-1}, 4, 3^{b_3+1}, \dots, 4, 3^{b_n})$$



At this point, we can apply the same extension to $b_3 + 1$, then to $b_4 + 1$, and so on. In this fashion, we can construct the full surface. $\square$

Our next goal is to decide how many of these extensions are actually distinct. To do so, we consider all closed paths in the hexagonal cake that only take the values 3 and 4.

Definition 8.3.25. *Let $(S, \widehat{\deg})$ be an extended SB–surface. If $\widehat{\deg}$ only takes the values 3 and 4, it is called a* ***cake boundary****.*

If the extended degree sequence of $(S, \widehat{\deg})$ is $(4, 3^{t_1}, 4, 3^{t_2}, \dots, 4, 3^{t_k})$, the ***type*** *of $\widehat{\deg}$ is the orbit of the dihedral group D_{2k} on $(t_1, t_2, \dots, t_k)$, under the action*

$$D_{2k} \times \mathbb{Z}^k \to \mathbb{Z}^k \qquad (g, (x_1, \dots, x_k)) \mapsto (x_{g^{-1}(1)}, \dots, x_{g^{-1}(x_k)}).$$

Any element of this orbit is a ***type representative****.*

In the proof of Theorem 8.3.24, we already showed how different boundary types can be related.

Remark 8.3.26. *Let $(S, \widehat{\deg})$ be an extended SB–surface such that $\widehat{\deg}$ is a cake boundary with type representative $(t_1, t_2, \dots, t_k)$, with $k \geq 2$.*

Then, there exists an extension $(U, \widehat{\deg}_U)$ such that $\widehat{\deg}_U$ is a cake boundary with type representative $(t_1 + 1, t_2 - 1, t_3 + 1, \dots, t_k)$.

We can extend the operation from Remark 8.3.26 into a group action of $\mathbb{Z}^k$.

Definition 8.3.27. *The group action of $\mathbb{Z}^k$ on $\mathbb{Z}^k$ via*

$$((a_1,\dots,a_k),\begin{pmatrix}x_1\\ \vdots\\ x_i\\ \vdots\\ x_k\end{pmatrix}) \mapsto \begin{pmatrix}x_1+a_n-a_1+a_2\\ \vdots\\ x_i+a_{i-1}-a_i+a_{i+1}\\ \vdots\\ x_k+a_{k-1}-a_k+a_1\end{pmatrix}$$

is called ***growth action****.*

The growth action is compatible with the action of the dihedral group from Definition 8.3.25. Thus, it induces an action on types.

Lemma 8.3.28. *Let B be the set of orbits of D_{2k} on $\mathbb{Z}^k$ and let $\mathcal{G} : \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{Z}^k$ be the growth action. Then, we have an action $\mathbb{Z}^k \times B \to B$,*

$$\mathbb{Z}^k \times B \to B \qquad\qquad (a, X) \mapsto \{\mathcal{G}(a,x) \mid x \in X\}.$$

Well–defined. Since the growth action and the dihedral group action

$$D_{2k} \times \mathbb{Z}^k \to \mathbb{Z}^k \qquad\qquad (g,(x_1,\dots,x_k)) \mapsto (x_{g^{-1}(1)},\dots,x_{g^{-1}(x_k)})$$

commute, the action is well–defined. □

We want to understand which cake boundary types result in the "same" extensions. Therefore, we are interested in the orbits of the growth action on these types.

So far, we know that applying the growth action does not change the infinite extension. We also want to show the other direction: If two cake boundaries construct the same infinite extension, their types lie in the same orbit of the growth action.

To do so, we introduce the concept of *height*. Informally, it measures the "distance" to the centre of the cake.

Definition 8.3.29. *Let $\uplus_{i=1}^k S_i$ be a hexagonal cake with base lengths $(b_1, b_2, \dots, b_k)$. A vertex $(x, y) \in V_i$ has* ***height*** *$x - b_i$.*

Well–defined. We have to show that the height is independent from the vertex representative. If $(h + b_i, y) \in V_i$ is equivalent to $(h + b_{i+1}, y^*) \in V_{i+1}$, the height remains invariant. □

Now, we can show that cake boundary types of cakes within the same infinite regular extension can be obtained by application of the growth action.

Lemma 8.3.30. *Let $(S, \widehat{\deg})$ be an extended SB–surface with negative boundary defect, such that $\widehat{\deg}$ is a cake boundary. Assume $(T, \widehat{\deg}_T)$ is an extension such that $\widehat{\deg}_T$ is also a cake boundary. Then, the types of these two cake boundaries lie in the same orbit of the action in Lemma 8.3.28.*

Proof. Let S^∞ be the infinite regular extension of $(S, \widehat{\deg})$ from Theorem 8.3.24. By Definition 8.2.3, there is a unique twilight morphism $\psi : T \to S^\infty$.

If there exists a cyclic interval $(v_1, v_2, \dots, v_k)$ in ∂T with

$$\widehat{\deg}_T(v_1) = \widehat{\deg}_T(v_k) = 4 \qquad\qquad \widehat{\deg}_T(v_2) = \cdots = \widehat{\deg}_T(v_{k-1}) = 3,$$

such that $\psi(v_i)$ is not a boundary vertex of the cake, we can reduce all of these vertices to obtain an extension with fewer faces. Since the reduction step changes the type of $\widehat{\deg}_T$ like the growth action, the claim follows. Therefore, we show the existence of such an interval.

For this, we insert a discussion about height. There are three different paths in a hexagonal slice that have external degree 3:

1. Only use edges in E_-: path $k \mapsto (x+k, y)$. The height increases at every step.

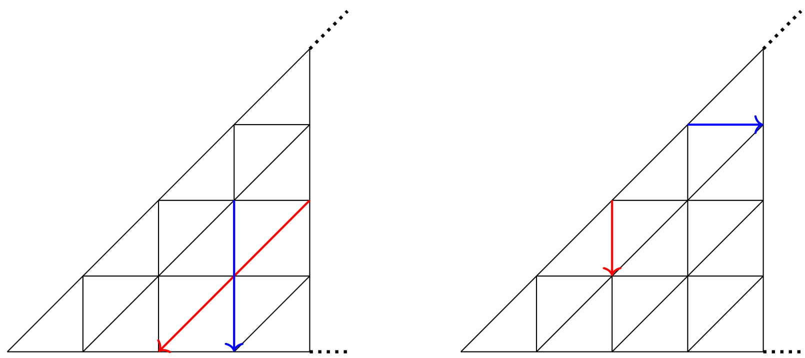
2. Only use edges in $E_/$: path $k \mapsto (x-k, y-k)$. The height decreases at every step.

3. Only use edges in $E_|$: path $k \mapsto (x, y-k)$. The height stays invariant.

We oriented the paths such that they move from the boundary $\{(k,k) \mid k \in \mathbb{Z}\}$ to the boundary $\{(k,0) \mid k \in \mathbb{Z}\}$. If we follow such a path, it can be modified by two scenarios:

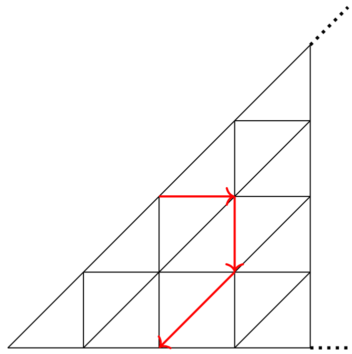
- If we change the slice, the path types change as follows:

$$E_/ \to E_| \to E_-$$



- If we encounter a vertex with external degree 4 that does not lie on a slice boundary, the path types change as follows:

$$E_- \to E_| \to E_/$$



- If we encounter a vertex with external degree 4 lying on a slice boundary, both of the previous path changes apply simultaneously. Thus, the path types remain unchanged.

We distinguish two cases.

1. If all vertices of ∂T with external degree 4 have height 0, there can be no path of the form $E_{/}$ (since the height cannot be decreased). There also cannot be a path of type E_{-} since it would be unbounded. Thus, all paths are of type $E_{|}$. If they change the slice, they would become a path of type E_{-}, whose height would rise indefinitely (since no vertex with external degree 4 could stop it). Thus, the vertex at the slice transition has to have external degree 4. In particular, ∂T runs along the boundary of the cake. By the construction of Theorem 8.3.24, this boundary is equal to ∂S. Thus, $(S, \widehat{\deg}_S) = (T, \widehat{\deg}_T)$.

2. Assume there is at least one vertex v with external degree 4 that has positive height. We want to show that the previous or the next vertex with external degree 4 has positive height. Assume the previous one has zero height. The path from the previous vertex to v can only have the types $E_{|}$ and E_{-} and it has to end with the type E_{-}. Therefore, the path to the next vertex starts with E_{-} or $E_{|}$. In either case, the height does not decrease. Thus, the height of the next vertex with external degree 4 is positive.

   By the previous argument, there is a cyclic interval $(v_1, v_2, \ldots, v_k)$ in ∂T with
   $$\widehat{\deg}_T(v_1) = \widehat{\deg}_T(v_k) = 4, \qquad \widehat{\deg}_T(v_2) = \cdots = \widehat{\deg}_T(v_{k-1}) = 3.$$
   If $\psi(v_i)$ is not a boundary vertex of the cake for every $2 \leq i < k$, we are finished. Otherwise, consider the map $\{1, \ldots, k\} \to \mathbb{Z}_{\geq 0}$, which assigns i the height of v_i. Since only slice transitions happen in this interval, and there are subsequences of decreasing and increasing height (the heights of v_1 and v_k are positive), there are $x, y \in \{2, \ldots, k-1\}$ such that:
   - $(v_1, \ldots, v_x)$ is a path of type $E_{/}$.
   - $(v_x, \ldots, v_y)$ is a path of type $E_{|}$.
   - $(v_y, \ldots, v_k)$ is a path of type E_{-}.

We follow ∂T further until the next vertex v_m of external degree 4, following the cyclic interval $(v_k, v_{k+1}, \ldots, v_m)$. By construction, the only possible path types are $E_|$ and E_-. Thus, the height of every v_i (with $k \leq i \leq m$) is positive. In this case, $(v_k, v_{k+1}, \ldots, v_m)$ is the desired cyclic interval. □

8.3.3 Boundary defect -1

In the previous Subsection 8.3.2, we constructed the infinite regular extension for extended SB–surfaces with negative boundary defect. In this subsection, we focus on those extended SB–surfaces with boundary defect -1. Our goal is to classify them.

In contrast to lower boundary defects (where we have to consider the growth action from Definition 8.3.27), there is essentially only one possibility to construct an infinite surface from one slice. This relies on an analysis of certain paths within a hexagonal slice.

Lemma 8.3.31. *Let H be the hexagonal cake consisting of a single slice and let P be a cyclic path in H of length L with external degree sequence $(4, 3, 3, \ldots, 3)$.*

Then, the $[(L, 0)]$ lies on the path and is the vertex with external degree 4.

Proof. The path has to go from (y, y) to $(0, y)$. If the vertex with external degree 4 is $[(0, y)]$, the only possible path is of type $E_|$.

Otherwise, any path uses two types of edges. Since it is not possible to use paths that increase *and* decrease height, there is no such path. □

This uniqueness–result allows us to define a surface invariant.

Definition 8.3.32. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect -1. Let $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$ be an element of the regular extension category with external degree sequence $(4, 3, \ldots, 3)$. Let f be the number of faces in T, and let L be the boundary length of T. The number $f - L^2$ is called the **staircase area** of $(S, \widehat{\deg})$.*

Well–defined. We have to show that the staircase area is independent from the choice of $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$. We can assume that the boundary of T is a path in a hexagonal cake consisting of one slice.

By Lemma 8.3.31, the path is unique. The number of faces between two paths of length L and $L - 1$ is $2L - 1$. Adding those up gives L^2 (for the difference to the hypothetical path of length 0). □

This gives a natural $\mathbb{Z}$–grading to the surfaces with boundary defect -1.

Corollary 8.3.33. *Since vertex splits increase the staircase area by 2, the growth–controlled SB–surfaces with boundary defect -1 are graded by their staircase area.*

8.3.4 Boundary defect -2

In Subsection 8.3.2, we constructed the infinite regular extension for extended SB–surfaces with negative boundary defect. In this subsection, we focus on those extended SB–surfaces with boundary defect -2. We would like to understand how many there actually are, i. e. to count the orbits of the growth action (compare Definition 8.3.27 and Lemma 8.3.28).

Remark 8.3.34. *The growth action $\mathcal{G} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2$ has three orbits, with representatives $(0, 0)$ and $(0, 1)$, and $(0, 2)$.*

Proof. Let $(x_1, x_2) \in \mathbb{Z}^2$ and $(a_1, a_2) \in \mathbb{Z}^2$. Then, the action of (a_1, a_2) on (x_1, x_2) is

$$(x_1 + 2a_2 - a_1, x_2 + 2a_1 - a_2).$$

We start by noting an invariant: The difference $x_2 - x_1$ modulo 3.

Next, we construct a canonical representative of the orbit. We try to find $(a_1, a_2) \in \mathbb{Z}^2$ such that $x_1 + 2a_2 - a_1 = 0$, which means $a_1 = x_1 + 2a_2$. Then, the action becomes

$$(0, x_2 + 2x_1 + 3a_2).$$

We can now choose a_2 to construct one of the three representatives. □

To transfer this statement to the action on cake boundary types, we need to do a bit more work:

Corollary 8.3.35. *The growth action on cake boundary types has two orbits. The first one contains all type representatives of the form (k, k) (for $k \in \mathbb{Z}_{\geq 0}$), the second one contains all type representatives of the form $(k, k + 1)$ (for $k \in \mathbb{Z}_{\geq 0}$).*

Proof. By the action of the dihedral group D_4, the classes with representatives $(0, 1)$ and $(0, 2)$ (from Remark 8.3.34) coincide.

Let $(x_1, x_2) \in (\mathbb{Z}_{\geq 0})^2$ be the type representative of a cake boundary, such that

$$\mathcal{G}((a_1, a_2), (x_1, x_2)) \in \{(0, 0), (0, 1)\},$$

where $\mathcal{G} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2$ is the growth action and $(a_1, a_2) \in \mathbb{Z}^2$. We show that there is a $(b_1, b_2) \in (\mathbb{Z}_{\geq 0})^2$ such that

$$\mathcal{G}((b_1, b_2), (x_1, x_2)) \in \{(k, k) \mid k \in \mathbb{Z}_{\geq 0}\} \cup \{(k, k + 1) \mid k \in \mathbb{Z}_{\geq 0}\}.$$

Let $k \in \mathbb{Z}_{\geq 0}$ such that $k + a_1 > 0$ and $k + a_2 > 0$, then

$$\mathcal{G}((a_1 + k, a_2 + k), (x_1, x_2)) = \mathcal{G}((a_1, a_2), (x_1, x_2)) + (k, k).$$

Therefore, $(a_1 + k, a_2 + k)$ is the desired (b_1, b_2). □

This allows us to choose some nice representatives for the two classes:

- Cakes with equal parameters
- Cakes where the parameters are 1 apart

For boundary defect 0 and -1 we defined the *staircase area* (Definition 8.3.17 and Definition 8.3.32), to give the SB–surfaces further structure. Extending this concept to boundary defect -2 requires a definition of canonical cake representative.

Definition 8.3.36. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect -2. A cake boundary is* ***balanced*** *if it has a type representative (k, k) or $(k, k+1)$ for a $k \in \mathbb{Z}$.*

We mention in passing that balanced cake boundaries correspond to (near) symmetric paths in [18].

Restricting the possible types to balanced ones makes the possible paths unique.

Lemma 8.3.37. *Let H be the hexagonal cake consisting of two slices with balanced cake boundary. Let P be a cyclic path in H that divides H into a finite component and another hexagonal cake with balanced cake boundary. Then, P only uses edges of type $E_|$.*

Proof. Since the path is cyclic, it ends at the same height at which it begins. By assumption, it passes over two slice transitions and two degree–4–vertices. Since edges in $E_|$ leave the height invariant, $E_/$ decrease it and E_- increase it, there are two options:

- All edges have the type $E_|$.
- There are edges in $E_/$ and E_-.

In the second case, the slice transitions and the degree–4–vertices cannot alternate (otherwise, only two edge types would be used). Thus, we have the subpath

$$E_- \to E_| \to E_/$$

for the two degree–4–vertices. The slice transitions complete it on both sides, and shift it to edges of type $E_|$. Since positive and negative heights have to balance out, the path P shifts the cake boundary from (a_1, a_2) to $(a_1 + 2j, a_2 - j)$ (like the growth action).

1. If $a_1 = a_2$, the boundary $(a_1 + 2j, a_2 - j)$ is not balanced except for $j = 0$.
2. If $a_1 = 1 + a_2$ or $a_1 + 1 = a_2$, the boundary $(a_1 + 2j, a_2 - j)$ is not balanced except for $j = 0$.

Thus, the only option is that all edge types lie in $E_|$ (this means that the degree–4–vertices coincide with the slice transitions). ☐

Like we did for Definition 8.3.32 in the case of boundary defect -1, we normalise the number of faces with respect to the length of a balanced boundary. Since the boundary lengths within a hexagonal cake change very predictably (compare Lemma 8.3.37), we can calculate the area for all balanced boundaries of higher length. We can also calculate the number of faces for all lower length, even for "hypothetical" cases like "0 length".

While this might not give a number of faces that is easy to interpret, it is a combinatorial invariant that does not depend on the concrete boundary length that is used in the computation of the number of faces.

Definition 8.3.38. *Let $(S, \widehat{\deg})$ be a growth–controlled extended SB–surface with boundary defect -2. Let $(T, \widehat{\deg}_T, (v_1, \ldots, v_n))$ be an element from the regular extension category whose external degree sequence is a balanced cake boundary of length L. Let f be the number of faces in T. The* ***staircase area*** *of $(S, \widehat{\deg})$ is*

- $f - \frac{L^2}{2}$*, if L is even.*
- $f - \frac{L^2-1}{2}$*, if L is odd.*

Well–defined. We start with the case that L is even. By Lemma 8.3.37, all other possible balanced boundaries have even length as well and are constructed from edges of type $E_|$. Thus, on a single slice they coincide with the paths from Lemma 8.3.31. Since L is even, both of these have length $l := \frac{L}{2}$. The number of faces between the paths of length L and $L-2$ is thus $2(2l-1)$. To compute the hypothetical difference to the path of length 0, we sum these up to obtain $2l^2 = \frac{L^2}{2}$.

Next, we consider L odd. Like in the previous case, we obtain two paths in a single slice, of length l and $l+1$ with $2l+1 = L$. Again, we compute the difference to a hypothetical path of length 1, to obtain

$$\begin{aligned}
\sum_{i=1}^{l}(2i-1) + \sum_{i=1}^{l}(2(i+1)-1) &= l^2 + \sum_{i=2}^{l+1}(2i-1) \\
&= l^2 + (l+1)^2 - 1 \\
&= 2l^2 + 2l \\
&= \frac{4l^2 + 4l}{2} \\
&= \frac{(2l+1)^2 - 1}{2} \\
&= \frac{L^2 - 1}{2}.
\end{aligned}$$

□

Vertex splits increase the staircase area by 2, which gives a natural $\mathbb{Z}$–grading.

Corollary 8.3.39. *Since vertex splits increase the staircase area by 2, the growth–controlled SB–surfaces with boundary defect -2 are graded by their staircase area.*

9 Geodesic Duality

In this chapter, we employ the formalism of Dress–surfaces (compare Section 2.6) to characterise all geodesic self–dual regular surfaces. *Geodesic duality* is the external surface symmetry called *opp* in Wilson's classification [71]. Core parts of this chapter are submitted in [11].

Using the correspondence to triangle subgroups from Subsection 4.3.2, we characterise the corresponding subgroups instead (Section 9.3). We obtain that all of these subgroups contain a particular normal subgroup.

Therefore, we can interpret a degree–d–surface as homomorphic image of a quotient of a triangle group (Section 9.4). To carry over the characterisation of subgroups corresponding to triangulations, we employ the voltage assignments from [3] (Section 9.5). We conclude with a characterisation of all geodesic self–dual degree–d–surfaces (Theorem 9.6.1), and give the full list of geodesic self–dual degree–d–surfaces for $d < 10$.

The approach to describe certain objects by subgroups of a "universal" group can be generalised quite far, compare [42].

9.1 Geodesic duality

Definition 2.6.1 allows the following duality: If $(C, \sigma_0, \sigma_1, \sigma_2)$ is a Dress–surface, then $(C, \sigma_0, \sigma_1, \sigma_0\sigma_2)$ is a Dress–surface as well (in [71, page 562], this operation is called *opp*).

Definition 9.1.1. *Let $S = (C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface. Its* ***geodesic dual*** *is the Dress–surface $(C, \sigma_0, \sigma_1, \sigma_0\sigma_2)$ and denoted by $S^{\#}$.*

Remark 9.1.2. *Let S be a Dress–surface, then $(S^{\#})^{\#} = S$, justifying the name duality.*

The geodesic dual of a Dress–surface can have very different properties than the original surface.

Example 9.1.3. *The geodesic dual of the tetrahedron in Section 2.6 is a projective plane, illustrated in Figure 9.1.*

Given a notion of duality, a common approach is to analyse self–dual objects. Here, we search for surfaces S that are isomorphic to their geodesic dual $S^{\#}$.

Definition 9.1.4. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface. It is called* ***geodesic self–dual*** *if it is isomorphic to its geodesic dual $(C, \sigma_0, \sigma_1, \sigma_0\sigma_2)$.*

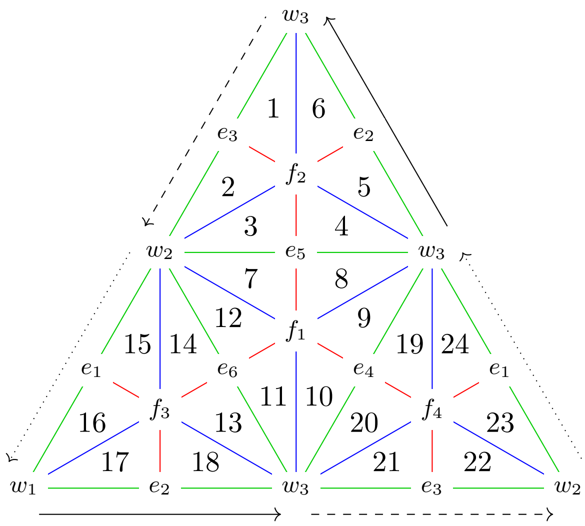


Figure 9.1: Geodesic dual of the tetrahedron

9.2 Geometric interpretation of geodesic duality

Definition 9.1.1 of geodesic duality in Section 9.1 seems very ungeometric. But, as the name suggests, there is a deeper geometric meaning there. Fix a chamber $c \in C$ and compare the actions of $\langle \sigma_1 \sigma_2 \rangle$ and $\langle \sigma_1 \sigma_0 \sigma_2 \rangle$ on c. Clearly, $\langle \sigma_1 \sigma_2 \rangle.c \subseteq \langle \sigma_1, \sigma_2 \rangle.c$, so each orbit of $\langle \sigma_1 \sigma_2 \rangle$ belongs to a unique vertex (compare Definition 2.6.3). If we consider the faces belonging to the chambers in $\langle \sigma_1 \sigma_2 \rangle.c$, we obtain all faces "around" a vertex – an umbrella. (illustrated in Figure 9.2).

The geometric meaning of $\langle \sigma_1 \sigma_0 \sigma_2 \rangle.c$ is not that easily apparent. Drawing the faces corresponding to the chambers in that orbit forms a "straight" strip of triangles (compare Figure 9.3). On a purely combinatorial level, these strips come closest to the notion of "straight lines". Therefore, we call these sets of faces **geodesics**.

Since geodesic duality exchanges the orbits $\langle \sigma_1 \sigma_2 \rangle.c$ and $\langle \sigma_1 \sigma_0 \sigma_2 \rangle.c$, it also exchanges umbrellas and geodesics in a surface. Heuristically, umbrellas are a local structure (to change an umbrella, you have to change the vertex it corresponds to or one of those adjacent to it), but geodesics show a global behaviour (if any vertex is modified, the set of geodesics may change drastically).

Therefore, geodesic duality seems to exchange some local and global properties in a given surface (and constructs a surface with inverted properties in the process). Since it relates very different surfaces (like tetrahedron and projective plane), one might hope to gain insight into one by analysing the other.

We mention in passing that the "zigzag–path" within a geodesic is sometimes referred to as *Petrie–polygon* (compare [24]).

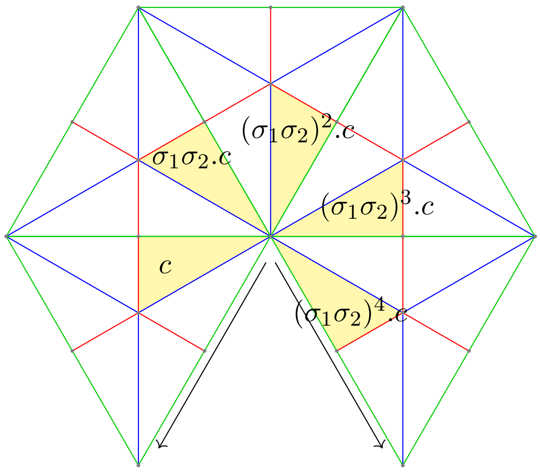


Figure 9.2: An umbrella

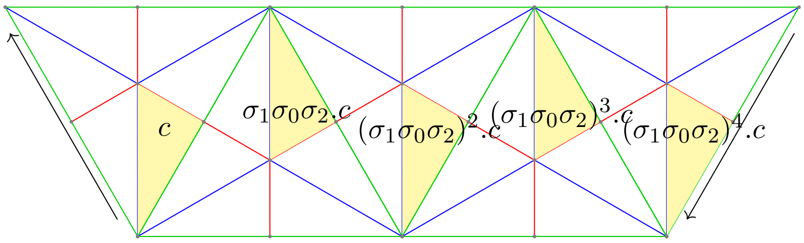


Figure 9.3: A geodesic

9.3 Geodesic triangle groups

In Subsection 4.3.2, we constructed a correspondence between degree–d–surfaces and certain subgroups of triangle groups, called *surface subgroups* (Definition 4.3.12).

In this section, we further restrict to geodesic self–dual degree–d–surfaces and characterise which surface subgroups correspond to them.

Lemma 9.3.1. *Let $U \leq T_d$ be a surface subgroup such that $(T_d/U, a, b, c)$ is a geodesic self–dual degree–d–surface. The normal closure $\langle\langle (bac)^d \rangle\rangle$ is contained in U.*

Proof. $(bc)^d$ acts trivially on each coset gU (for $g \in T_d$). By self–duality, $(bac)^d$ also acts trivially on gU. In other words, $(bac)^d gU = gU$, or $g^{-1}(bac)^d g \in U$ for all $g \in T_d$. □

Since the normal subgroup $\langle\langle (bac)^d \rangle\rangle$ is always contained in surface subgroups of geodesic self–dual degree–d–surfaces, we can factor it out.

Definition 9.3.2. *For every $d \in \mathbb{N}$, the* ***geodesic triangle group*** *is defined as*

$$H_d := \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^2, (bc)^d, (bac)^d \rangle. \tag{9.1}$$

These groups were considered in greater generality in [24, Subsection 8.6]. In our language, they did not enforce geodesic self–duality, and their notation for H_d is $\{d,3\}_d$. They analysed a few of these groups but did not attempt a classification.

Remark 9.3.3. *There is a lattice isomorphism $\{\langle\!\langle (bac)^d \rangle\!\rangle \leq U \leq T_d\} \to \{V \leq H_d\}$. Since $\langle\!\langle (bac)^d \rangle\!\rangle$ acts trivially on T_d/U, it is sufficient to consider the action of $T_d/\langle\!\langle (bac)^d \rangle\!\rangle$ on T_d/U. This action is equivariant to the action of H_d on H_d/V (with $V = U/\langle\!\langle (bac)^d \rangle\!\rangle$). Therefore, $(T_d/U, a, b, c)$ and $(H_d/V, a, b, c)$ describe the same degree–d–surface.*

Geodesic duality can be formulated on the level of surface subgroups.

Definition 9.3.4. *The* ***geodesic automorphism*** $\# : H_d \to H_d$, *is defined by*

$$a \mapsto a \qquad\qquad b \mapsto b \qquad\qquad c \mapsto ac.$$

For $g \in H_d$ and $V \leq H_d$, we employ the notation $g^\# := \#(g)$ and $V^\# := \{g^\# \mid g \in V\}$.

Well–defined. Let F be the free group generated by $\bar{a}$, $\bar{b}$, and $\bar{c}$. Then

$$\bar{\#} : F \to H_d, \qquad \bar{a} \mapsto a, \quad \bar{b} \mapsto b, \quad \bar{c} \mapsto ac$$

is a well–defined group homomorphism. We consider its kernel. Since $\bar{\#}(\bar{a}) = a$ and $\bar{\#}(\bar{b}) = b$, we immediately get $\langle\!\langle \bar{a}^2, \bar{b}^2, (\bar{a}\bar{b})^3 \rangle\!\rangle \leq \ker \bar{\#}$.

$\bar{c}^2$ and $(\bar{a}\bar{c})^2$ are mapped to $(ac)^2 = 1$ and $c^2 = 1$, so both lie in the kernel of $\bar{\#}$. An analogous argument shows that $(\bar{b}\bar{c})^d$ and $(\bar{b}\bar{a}\bar{c})^d$ lie in $\ker \bar{\#}$. Thus, $\bar{\#}$ factors over the normal subgroup generated by these relations (which gives $\#$). □

The geodesic automorphism allows us to transfer the notion of geodesic duality from the level of Dress–surfaces to the subgroups of the geodesic triangle group.

Proposition 9.3.5. *Let $V \leq H_d$ such that $S = (H_d/V, a, b, c)$ is a degree–d–surface. Then its geodesic dual is given by $(H_d/V^\#, a, b, c)$.*

In particular, S is geodesic self–dual if and only if $V^\#$ is conjugate to V in H_d.

Proof. The geodesic dual of S is $(H_d/V, a, b, ac)$. This corresponds to the action of H_d on the cosets of V in H_d via

$$\varphi : H_d \times H_d/V \to H_d/V, \qquad (h, tV) \mapsto h^\# tV.$$

We want to show that this action is equivariant to

$$\psi : H_d \times H_d/V^\# \to H_d/V^\#, \qquad (h, tV^\#) \mapsto htV^\#.$$

Since $\#$ is an automorphism of H_d, we have a bijection

$$\# : H_d/V \to H_d/V^\#, \qquad tV \mapsto t^\# V^\#.$$

By Definition 4.3.4, we have to show that $\varphi(h, tV)^\# = \psi(h, (tV)^\#)$:

$$\varphi(h, tV)^\# = (h^\# tV)^\# = h(tV)^\# = \psi(h, (tV)^\#).$$

Therefore, the geodesic dual is given by $(H_d/V^\#, a, b, c)$. This degree–d–surface is isomorphic to $(H_d/V, a, b, c)$ if and only if $V^\#$ and V are conjugate in H_d. □

Finally, we can characterise self–dual degree–d–surfaces group–theoretically.

Corollary 9.3.6. *Let $U \leq T_d$. Then, $(T_d/U, a, b, c)$ is a geodesic self–dual degree–d–surface if and only if*

- $gUg^{-1} \cap X = \{1\}$ *for all* $g \in T_d$ *and* $X \in \{\langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle\}$.
- $\langle\langle (bac)^d \rangle\rangle \leq U$.
- $(U/\langle\langle (bac)^d \rangle\rangle)^\#$ *is conjugate to* $U/\langle\langle (bac)^d \rangle\rangle$ *in* H_d.

Furthermore, every geodesic self–dual degree–d–surface has this form.

Proof. From Corollary 4.3.11 and Definition 4.3.12 we deduce that every degree–d–surface corresponds to a surface subgroup $U \leq T_d$ and can be represented as $S = (T_d/U, a, b, c)$. Surface subgroups are characterised in Lemma 4.3.14. This gives the first condition of the statement.

If S is geodesic self–dual, Lemma 9.3.1 gives the necessary condition $\langle\langle (bac)^d \rangle\rangle \leq U$. This condition allows the reduction to $V := U/\langle\langle (bac)^d \rangle\rangle$ in H_d by Remark 9.3.3.

Then, Proposition 9.3.5 gives the final condition of the statement. □

Since every element of the dihedral group $\langle x, y \mid x^2, y^2, (xy)^k\rangle$ is conjugate to x, y or $(xy)^m$ (with m dividing k), we can replace the sets X from Corollary 9.3.6 by

$$X \in \{\langle a\rangle, \langle b\rangle, \langle c\rangle, \langle ab\rangle, \langle ac\rangle, \langle bc\rangle\}. \tag{9.2}$$

9.4 Reduction to geodesic triangle groups

Corollary 9.3.6 characterises geodesic self–dual degree–d–surfaces by considering both groups, T_d and H_d. We would like to have a characterisation in which only H_d appears, since H_d is often smaller than T_d.

Remark 9.4.1. *The groups T_d and H_d have the following orders:*

d	1	2	3	4	5	6	7	8	9	≥ 10
$\lvert T_d\rvert$	2	12	24	48	120	∞	∞	∞	∞	∞
$\lvert H_d\rvert$	1	4	1	4	60	108	1	672	3420	∞

Proof. The finiteness results can be calculated very easily in GAP ([33]). They can also be found in [24, Table 8], if one uses the notation $H_d = \{d, 3\}_d$.

For H_d, Edjvet and Juhász show all of the results in [30]. In comparison to our notation, the roles of b and c are interchanged. Therefore, the parameters in their paper are set as follows: $m := 3$ and $n = p := d$. □

We conclude that there are no geodesic self–dual degree–d–surfaces for $d \in \{3, 4, 7\}$.

Corollary 9.4.2. *Let $U \leq T_d$ and $\langle\langle (bac)^d \rangle\rangle \cap \langle bc\rangle \neq \{1\}$. Then, $(T_d/U, a, b, c)$ is not a geodesic self–dual degree–d–surface.*

In particular, there is no geodesic self–dual degree–d–surface for $d \in \{3, 4, 7\}$.

Proof. If $(T_d/U, a, b, c)$ was a geodesic self–dual degree–d–surface, Lemma 9.3.1 would give $\langle\langle (bac)^d \rangle\rangle \leq U$. But $U \cap \langle bc \rangle \geq \langle\langle (bac)^d \rangle\rangle \cap \langle bc \rangle \neq \{1\}$, which contradicts the characterisation of surface subgroups in Lemma 4.3.14.

Since $H_3 = H_7 = \{1\}$, we have $\langle\langle (bac)^d \rangle\rangle = T_d$ in these cases. For H_4, it can be checked (either with GAP or with a calculation like in [15, Section 3.3]) that $c(bac)^4cb(bac)^4b = (cb)^2$. Explicitly:

$$
\begin{aligned}
c(bac)^4cb(bac)^4b &= c(bac)^3ba\underline{ccbb}ac(bac)^3b \\
&= c(bac)^2bacb\underline{aa}cb\underline{ac}(bac)^2b \\
&= c(bac)^2ba\underline{cbcbc}a(bac)^2b \\
&= c(bac)^1bac\underline{bab}c\underline{bab}ac(bac)^1b \\
&= c(bac)^1b\underline{aca}b\underline{aca}b\underline{aa}c(bac)^1b \\
&= cba\underline{cbcbcbcb}acb \\
&= cb\underline{aa}cb \\
&= (cb)^2
\end{aligned}
$$

□

We would like to replace $gUg^{-1} \cap X = \{1\}$ for $g \in T_d$ by $gVg^{-1} \cap X = \{1\}$ for $g \in H_d$.

Lemma 9.4.3. *Let G be a group, $W, X \leq G$ and $N \trianglelefteq G$ with $N \leq W$ and $X \cap N = \{1\}$. Then $W \cap X \cong W/N \cap XN/N$.*

Proof. The result follows from the homomorphism theorems:

$$
\begin{aligned}
W \cap X \cong (W \cap X)/(W \cap X \cap N) &\cong (W \cap X)N/N \\
&= WN/N \cap XN/N = W/N \cap XN/N.
\end{aligned}
$$

□

To apply this lemma to $G = T_d$, $W = gUg^{-1}$, $X = X$, and $N = \langle\langle (bac)^d \rangle\rangle$, we need to show that $\langle\langle (bac)^d \rangle\rangle \cap X = \{1\}$ for all X from Equation (9.2). This is easy for $X \neq \langle bc \rangle$.

Lemma 9.4.4. *$\langle\langle (bac)^d \rangle\rangle \cap X = \{1\}$ for $X \in \{\langle a \rangle, \langle b \rangle, \langle c \rangle, \langle ab \rangle, \langle ac \rangle\}$ and $d \geq 5$ with $d \neq 7$.*

Proof. If $a \in \langle\langle (bac)^d \rangle\rangle$, we also have $\langle\langle a \rangle\rangle \leq \langle\langle (bac)^d \rangle\rangle$. In particular,

$$
H_d \cong T_d/\langle\langle (bac)^d \rangle\rangle \cong (T_d/\langle\langle a \rangle\rangle)/(\langle\langle (bac)^d \rangle\rangle/\langle\langle a \rangle\rangle).
$$

Since $T_d/\langle\langle a \rangle\rangle = \langle b, c \mid b^2, c^2, (bc)^d \rangle$ is a dihedral group of order $2d$, we conclude $|H_d| < 2d$. By Remark 9.4.1, this cannot happen for $d \geq 5$ and $d \neq 7$. Similar arguments apply to b and c. We can apply the same argument to ab and ac. We get $T_d/\langle\langle ab \rangle\rangle \cong D_4$ and $T_d/\langle\langle ac \rangle\rangle \cong D_6$, so $\langle\langle (bac)^d \rangle\rangle \cap \langle ab \rangle$ and $\langle\langle (bac)^d \rangle\rangle \cap \langle ac \rangle$ are trivial as well. □

At this point, we can reformulate the characterisation from Corollary 9.3.6.

Corollary 9.4.5. *Let $V \leq H_d$ with $d \geq 5$ and $d \neq 7$. Then, $(H_d/V, a, b, c)$ is a geodesic self–dual degree–d–surface if and only if*

- $gVg^{-1} \cap X = \{1\}$ *for* $X \in \{\langle a\rangle, \langle c\rangle, \langle ab\rangle, \langle ac\rangle, \langle bc\rangle\}$ *and all* $g \in H_d$.
- $V^{\#}$ *is conjugate to* V.
- $\langle\!\langle (bac)^d \rangle\!\rangle \cap \langle bc\rangle = \{1\}$ *in* T_d.

Furthermore, every geodesic self–dual degree–d–surface has this form.

Proof. Let $(H_d/V, a, b, c)$ be geodesic self–dual degree–d–surface. Then, there is a surface subgroup $U \leq T_d$ with $U/\langle\!\langle (bac)^d \rangle\!\rangle = V$. By Corollary 9.3.6, $gUg^{-1} \cap X = \{1\}$ for all X in the list (9.2). By assumption and Lemma 9.4.4, we can apply Lemma 9.4.3 to conclude $gVg^{-1} \cap X = \{1\}$.

Conversely, there is an $U \leq T_d$ with $U/\langle\!\langle (bac)^d \rangle\!\rangle = V$. Since $(ab)a(ab)^{-1} = b$, we also have $gVg^{-1} \cap \langle b\rangle = \{1\}$. Since $V \cap \langle ac\rangle \cong (V \cap \langle ac\rangle)^{\#} = V^{\#} \cap \langle c\rangle = hVh^{-1} \cap \langle c\rangle$, for some $h \in H_d$, this intersection is also trivial. Applying Lemma 9.4.3 shows that the conditions of Corollary 9.3.6 are fulfilled. $\square$

9.5 Uncollapsed geodesic triangle groups

The characterisation of geodesic self–dual degree–d–surfaces in Corollary 9.4.5 contains the assumption $\langle\!\langle (bac)^d \rangle\!\rangle \cap \langle bc\rangle = \{1\}$. In this section, we show that this condition is not necessary. We start by rewriting it.

Remark 9.5.1. *Since # is an automorphism of* H_d, $(bc)^k = 1$ *if and only if* $(bac)^k = 1$.

Lemma 9.5.2. *In the triangle group* T_d*, we have* $\langle\!\langle (bac)^d \rangle\!\rangle \cap \langle bc\rangle = \langle (bc)^k\rangle$ *for* $1 \leq k \leq d$ *if and only if* $H_d = H_k$.

Proof. Suppose first that $\langle\!\langle (bac)^d \rangle\!\rangle \cap \langle bc\rangle = \langle (bc)^k\rangle$. Clearly, k divides d. Then:

$$\begin{aligned}
H_d &= T_d/\langle\!\langle (bac)^d \rangle\!\rangle \\
&= T_d/\langle\!\langle (bc)^k, (bac)^d \rangle\!\rangle \\
&= \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^2, (bc)^d, (bc)^k, (bac)^d\rangle \\
&= \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^2, (bc)^k, (bac)^d\rangle.
\end{aligned}$$

Since $(bc)^k = 1$, Remark 9.5.1 implies $(bac)^k = 1$ as well.

For the other direction, we note that $T_d \neq T_k$ since the factor groups with respect to $\langle\!\langle a \rangle\!\rangle$ are dihedral groups of different orders. Therefore, $H_d = H_k$ implies that $(bc)^k \in \langle\!\langle (bac)^d \rangle\!\rangle$ and $(bac)^k \in \langle\!\langle (bac)^d \rangle\!\rangle$. $\square$

This motivates the following definition.

Definition 9.5.3. H_d *is* ***uncollapsed*** *if* $H_d \neq H_k$ *for all* $1 \leq k \leq d$.

We want to show that H_d is uncollapsed if $d \geq 5$ and $d \neq 7$.

Lemma 9.5.4. H_d *is uncollapsed if and only if* $H_{\frac{d}{p}} \neq H_d$ *for all primes* p *dividing* d.

Proof. If $H_k = H_d$ with $\frac{d}{k}$ not prime, there is a prime p dividing this fraction, such that

$$H_{\frac{d}{p}} = H_d / \langle\!\langle (bc)^{\frac{d}{p}}, (bac)^{\frac{d}{p}} \rangle\!\rangle = H_k / \langle\!\langle (bc)^{\frac{d}{p}}, (bac)^{\frac{d}{p}} \rangle\!\rangle = H_k = H_d. \qquad \square$$

Corollary 9.5.5. *Let $p \notin \{3, 7\}$ be a prime. Then H_p is uncollapsed. Furthermore, H_d is uncollapsed for $d \in \{2, 6, 12, 15, 21, 35, 49\}$.*

Proof. Apply Lemma 9.5.4. We have $\{1\} = H_1 = H_p$ if and only if $p \in \{3, 7\}$.

If H_d is finite, we can inspect the table from Remark 9.4.1 to see whether it is uncollapsed. If H_d is infinite, but for every prime p dividing d, the group $H_{\frac{d}{p}}$ is finite, then H_d has to be uncollapsed. $\square$

Lemma 9.5.6. *Let p be an odd prime. Then H_{2p} is uncollapsed.*

Proof. By Lemma 9.5.4, we only need to consider H_2 and H_p. By Remark 9.4.1, $|H_{2p}| \neq 2$.

Consider the map $H_d \to \{\pm 1\}$ that maps a, b, and c all to -1. It is only well–defined for even d, thus $H_{2p} \neq H_p$. $\square$

Theorem 9.5.7. *H_{2^n} is uncollapsed ($n \geq 3$). H_{3^n} is uncollapsed ($n \geq 2$). H_{5^n} is uncollapsed ($n \geq 1$).*

Proof. This can be shown by a lengthy calculation[1] in GAP ([33]). We compute a presentation of the subgroups $\langle\!\langle (bac)^8 \rangle\!\rangle \leq H_{2^n}$, $\langle\!\langle (bac)^9 \rangle\!\rangle \leq H_{3^n}$, and $\langle\!\langle (bac)^5 \rangle\!\rangle \leq H_{5^n}$, by using the subgroup presentation algorithm in [37, Section 2.5]. Then, we calculate the abelian invariants of this subgroups to distinguish the groups.

Formally, we want to construct a presentation for $N_{p^k} := \langle\!\langle (bac)^{p^k} \rangle\!\rangle$ in H_{p^n} ($p^k \in \{2^3, 3^2, 5^1\}$) and compute the derived subgroup of N_p. Our notation for the subgroup presentation algorithm follows the *Handbook of Computational Group Theory* [37].

Since we have to perform the algorithm partially by hand, we start with defining some preliminary algorithms:

```
# Given a homomorphism from a free group into a factor group,
# compute the Schreier transversal of its kernel
SchreierTransversal := function( hom )
    local trans, todo, imageList, gens, word, g, free;

    free := Source(hom);
    trans := [];
    todo := [ One(free) ];
    gens := GeneratorsOfGroup(free);
    imageList := [];
    while not IsEmpty(todo) do;
        word := Remove(todo, 1);
```

[1] Code: `https://markusbaumeister.github.io/code/UncollapsedGeodesicTriangleGroups.g`.

```
        # Check if this element already was found
        if not Image(hom, word) in imageList then
            Add(trans, word);
            Add(imageList, Image(hom, word));
            for g in gens do
                Add(todo, word*g);
            od;
        fi;
    od;

    return trans;
end;

# Given an element in the free group, the homomorphism into
# a factor group and the transversal of the kernel, return
# the representative of the element in the transversal
SchreierRep := function( el, hom, transversal )
    local test;

    for test in transversal do
        if Image(hom, test) = Image(hom, el) then
            return test;
        fi;
    od;

    Error("No representative found.");
end;

# Given a homomorphism, a transversal of its kernel and two
# elements t,x, compute their Schreier generator
SchreierGenerator := function( hom, transversal, t, x )
    return t * x * SchreierRep( t*x, hom, transversal )^(-1);
end;
```

We set the prime numbers by choosing one of these three options:

```
# p := 2; pk := p^3;
# p := 3; pk := p^2;
# p := 5; pk := p^1;
```

Our approach makes use of the following epimorphisms:

$$\langle a, b, c \rangle \to \langle a, b, c | a^2, b^2, c^2, (ab)^3, (ac)^2 \rangle \to H_{p^n}$$

```
F := FreeGroup( "a", "b", "c" );
generalRel := [ F.a^2, F.b^2, F.c^2, (F.a*F.c)^2, (F.a*F.b)^3 ];
```

```
Hgen := F/generalRel;
# Map from F to Hgen
homGen := GroupHomomorphismByImages( F, Hgen,
                [F.a,F.b,F.c], [Hgen.1,Hgen.2,Hgen.3]);
```

In particular, we can consider H_{p^k} as a factor group of $\langle a,b,c|a^2,b^2,c^2,(ab)^3,(ac)^2\rangle$.

```
Ngen := NormalClosure( Hgen,
    Group([ (Hgen.2*Hgen.3)^pk, (Hgen.2*Hgen.1*Hgen.3)^pk ]) );
# Map from Hgen to N_{p^k}
homNgen := NaturalHomomorphismByNormalSubgroup(Hgen, Ngen);
# Map from F to N_{p^k}
homNinfree := CompositionMapping2( homNgen, homGen );
Hpk := Image(homNgen);
```

The Schreier–transversal has to be computed with respect to the free group.

```
transversal := SchreierTransversal( homNinfree );
```

For the Schreier–transversal T and the generators $X = \{a,b,c\}$, we have to compute the Schreier–generators

$$\{tx\overline{tx}^{-1}|t \in T, x \in X, tx \neq_{\langle a,b,c\rangle} \overline{tx}\}$$

Computationally, we store them as triples $(t,x,tx\overline{tx}^{-1})$.

```
schreierGens := [];
for t in transversal do
    for x in [F.a,F.b,F.c] do
        Add(schreierGens,
            [t,x,SchreierGenerator(homNinfree, transversal, t,x)]);
    od;
od;
```

The Schreier–generators Y generate N_{p^k}. By [37, Theorem 2.62], $N_{p^k} \cong \langle Y|S\rangle$ with $S = \{\rho(twt^(-1))|t \in T, w \in R\}$ (here, R are the relations of H_{p^n} and ρ rewrites into the Schreier–generators Y).

We split the relations R into two sets: the general relations $\{a^2,b^2,c^2,(ab)^3,(ac)^2\}$ and the prime–specific relations $\{(bc)^{p^n},(bac)^{p^n}\}$. First, we rewrite the general relations in the Schreier–generators.

```
simpleRelations := [];
for rel in generalRel do
    for t in transversal do
        Add( simpleRelations, t*rel*t^(-1) );
    od;
od;

# Rewrite the simple relations in the Schreier-generators
```

```
subgroup := Group( List(schreierGens, i->i[3]) );
rewriteHom := EpimorphismFromFreeGroup( subgroup );
G := Source(rewriteHom);

newRels := [];
for i in [1..Length(schreierGens)] do
    if schreierGens[i][3] = Identity(F) then
        Add(newRels, G.(i));
    fi;
od;
for rel in simpleRelations do
    Add(newRels, PreImagesRepresentative(rewriteHom, rel));
od;
```

At this point, there are a lot of generators and a lot of relations. We simplify the presentation $\langle Y|S_{gen}\rangle$.

```
pres := PresentationFpGroup(G/newRels);
TzGoGo(pres);
```

For all $p \in \{2,3,5\}$, this results in a free group without relations. Furthermore, its generators are a subset of the Schreier–generators. We use a little hack to obtain this subset.

```
remainingIndices := [];
for gen in GeneratorsOfPresentation(pres) do
    # This is a hack to obtain the number of the generator
    str := ShallowCopy( String(gen) );
    Remove(str,1);
    pos := Int(str);
    Add( remainingIndices, pos );
od;
```

To rewrite the prime–specific relations, we want to use only this restricted subset of Schreier–generators. We need another rewrite homomorphism.

```
minSubgroup := Group( List( remainingIndices,
    i -> Image(homGen, schreierGens[i][3])  ) ); # subgroup of Hgen
minRewriteHom := EpimorphismFromFreeGroup( minSubgroup );
a := Hgen.1;
b := Hgen.2;
c := Hgen.3;
```

At this point, we face a problem: $t(bc)^{p^n}t^{-1}$ can't be rewritten for general n. Fortunately, this is not necessary. We can reformulate

$$t(bc)^{p^n}t^{-1} = (t(bc)^{p^k}t^{-1})^{p^{n-k}}.$$

Then it is sufficient to rewrite $t(bc)^{p^k}t^{-1}$.

```
bcList := List( transversal,
    t -> Image(homGen, t)^(-1) * (b*c)^pk * Image(homGen, t) );
bacList := List( transversal,
    t -> Image(homGen, t)^(-1) * (b*a*c)^pk * Image(homGen, t) );
paramRels := Set(Concatenation(bcList, bacList));

trueRels := List(paramRels,
    rel -> PreImagesRepresentative(minRewriteHom, rel));
minRels := [];
for t in trueRels do
    found := false;
    for m in minRels do
        if t = m or t = m^(-1) then
            found := true;
        fi;
    od;
    if not found then
        Add(minRels, t);
    fi;
od;
```

Each element r of `minRels` corresponds to the relation $r^{p^{n-k}}$.

Without further simplification of the presentation we compute the abelian invariants of N_{p^k}. By [37, Section 9.2], we can compute these invariants by writing the relations into a matrix (additively) and computing the Hermite normal form. Since all relations have the same exponent, we can rewrite this matrix as $p^{n-k}M$, where M is the matrix formed from all r in `minRels`.

In particular, it is sufficient to compute the abelian invariants of $\langle Y | S_{gen}, \texttt{minRels}^p \rangle$ and show that these are not all 0.

```
coreQuotient := Source(minRewriteHom)/List(minRels, r -> r^p);
AbelianInvariants(coreQuotient);
```

For $p = 5$, the largest abelian factor group is C_5^6, for $p = 3$ it is C_3^{286} and for $p = 2$ it is C_2^{57}. □

9.5.1 Voltage assignments

In Section 9.5 we introduced the notion of *uncollapsed* to capture a necessary condition for the characterisation of geodesic self–dual surface in Corollary 9.4.5.

In this subsection, we show that H_{p^n} and H_{4p} are uncollapsed. To achieve this, we use *corner voltage assignments* to construct appropriate surface coverings. The presentation of this theory follows [3].

Definition 9.5.8. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface and B be a group. A map $v : C \to B$ is called* ***corner voltage assignment****, if $v(\sigma_1.c) = v(c)^{-1}$ holds for all $c \in C$. In this scenario, B is called the* ***voltage group****.*

The formalism of voltage assignments is very effective to construct covering surfaces.

Definition 9.5.9. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a Dress–surface with corner voltage assignment $v : C \to B$. The* ***lift*** *of $(C, \sigma_0, \sigma_1, \sigma_2)$ with respect to v is the quadruple $(C \times B, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2)$, with*

$$\hat{\sigma}_0.(c, g) := (\sigma_0.c, g), \qquad \hat{\sigma}_1.(c, g) := (\sigma_1.c, v(c)g), \qquad \hat{\sigma}_2.(c, g) := (\sigma_2.c, g),$$

In general, the lift does not define a surface in the sense of Definition 2.6.1 and Definition 4.3.1. There are two possible reasons: The orbits of $\langle \hat{\sigma}_0, \hat{\sigma}_1 \rangle$ might have an order larger than 3, and the group $\langle \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2 \rangle$ might not act transitively on $C \times B$. The transitivity can be achieved by restriction to a single orbit.

The orbit lengths of $\langle \hat{\sigma}_0, \hat{\sigma}_1 \rangle$ can be controlled by an additional condition.

Remark 9.5.10. *Let $(C, \sigma_0, \sigma_1, \sigma_2)$ be a surface with corner voltage assignment $v : C \to B$ and lift $(C \times B, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2)$. Then,*

1. *$\hat{\sigma}_0$, $\hat{\sigma}_1$, and $\hat{\sigma}_2$ are involutions without fixed points on $C \times B$.*
2. *$\langle \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2 \rangle$ acts transitively on each of its orbits.*
3. *$\hat{\sigma}_0\hat{\sigma}_1$ consists only of 3–cycles if and only if $v(\sigma_1\sigma_0.c)v(\sigma_0\sigma_1.c)v(c) = 1$ for all $c \in C$.*
4. *$\hat{\sigma}_0\hat{\sigma}_2$ consists only of 2–cycles.*

Proof. Most properties follow from the corresponding properties for surfaces (compare Definition 2.6.1 and Definition 4.3.1). We compute $(\hat{\sigma}_0\hat{\sigma}_1)^3.(c, g)$ for $(c, g) \in C \times B$:

$$(\hat{\sigma}_0\hat{\sigma}_1)^2.(\sigma_0\sigma_1 c, v(c)g) = (\hat{\sigma}_0\hat{\sigma}_1).(\hat{\sigma}_1\hat{\sigma}_0 c, v(\sigma_0\sigma_1.c)v(c)g) = (c, v(\sigma_1\sigma_0.c)v(\sigma_0\sigma_1.c)v(c)g)$$

Therefore, the product condition is equivalent to $(\hat{\sigma}_0\hat{\sigma}_1)^3 = 1$. If there is an element (c, g) that does not lie in a 3–cycle of $\hat{\sigma}_0\hat{\sigma}_1$, it has to be fixed by it. But then, c would have to be fixed by $\sigma_0\sigma_1$, contradicting that we started with a Dress–surface. □

Before we can construct appropriate lifts, we need to prove a few technical lemmas.

Lemma 9.5.11. *Let H_d be infinite. Then, there is no k with $\gcd(k, d) = 1$ such that $(bc)^k \in \langle a, b \rangle$ or $(bac)^k \in \langle a, b \rangle$.*

Proof. Without loss of generality, only consider bc. Since $(bc)^d = 1$, we can apply the `Euclidean` algorithm to deduce $bc \in \langle a, b \rangle$. A short computation in GAP ([33]) shows that H_d has to be finite in this case. □

Lemma 9.5.12. *Let $p \geq 5$ be a prime and H_{p^n} be infinite and uncollapsed. If either $(bc)^k \in \langle a, b \rangle$ or $(bac)^k \in \langle a, b \rangle$ holds, then k is a multiple of p^n.*

Proof. By Lemma 9.5.11, k cannot be coprime to p^n. If k is not a multiple of p^n, we can reduce to the case $k = p^m$ with $0 < m < n$. In this case, $1 = (bc)^{p^n} = ((bc)^{p^m})^{p^{n-m}} = x^{p^{n-m}}$ for some $x \in \langle a, b \rangle$.

Since $p \geq 5$, the element x cannot have order 2 or 3. The only remaining element in $\langle a, b \rangle$ is 1. But $x = 1$ would imply $H_{p^n} = H_{p^m}$ (by Remark 9.5.1), contradicting H_{p^n} being uncollapsed. □

Lemma 9.5.13. *Let p be a prime and H_{2p} be infinite and uncollapsed. If either $(bc)^k \in \langle a, b \rangle$ or $(bac)^k \in \langle a, b \rangle$ holds, then k is a multiple of $2p$.*

Proof. By Lemma 9.5.11, k cannot be coprime to $2p$. If k is not a multiple of $2p$, we can reduce to the cases $k = 2$ or $k = p$. For $k = 2$, it is easy to check with GAP ([33]) that every case of $(bc)^2 \in \langle a, b \rangle$ implies the finiteness of H_{2p}.

For $k = p$, we have $1 = (bc)^{2p} = ((bc)^p)^2$. Therefore, $(bc)^p = 1$ or has order 2. The first case is impossible since H_{2p} is uncollapsed. The second one implies $(bc)^p \in \{a, b, aba\}$. To analyse these cases, we use the equality $(bc)^p b (bc)^p = b(cb)^p (bc)^p = b$:

$$
\begin{aligned}
(bc)^p &= a & &\text{implies} & 1 &= (ab)^3 = ((bc)^p b)^3 = b^2 (bc)^p b = ab, \\
(bc)^p &= aba & &\text{implies} & 1 &= (ab)^3 = (bc)^p b (bc)^p a = ba.
\end{aligned}
$$

In this case, H_d is finite. From $(bc)^p = b$ we can deduce $(bc)^{p-1} = c$, from which $(bc)^{p-2} = b$ follows. Inductively, either $b = 1$ or $c = 1$ holds, then H_{2p} is finite. □

Proposition 9.5.14. *Let H_d be uncollapsed such that $(bc)^k \in \langle a, b \rangle$ or $(bac)^k \in \langle a, b, \rangle$ is only possible if k is a multiple of d. For any prime p, there exists a degree–dp–surface $(D, \hat{\sigma_0}, \hat{\sigma}_1, \hat{\sigma}_2)$ such that $\hat{\sigma}_1 \hat{\sigma}_0 \hat{\sigma}_2$ consists only of dp-cycles. In particular, $H_d \neq H_{dp}$.*

Proof. To construct the surface, we start with the geodesic self–dual degree–d–surface $(H_d/\{1\}, a, b, c)$. Let F be the set of $\langle a, b \rangle$–orbits (the faces). Choose one element f from each orbit to represent the orbit as $\langle a, b \rangle.f$. Define the voltage group

$$
B := \begin{cases} (\mathbb{Z}/p\mathbb{Z})^F & p \neq 2 \\ (V_4)^F & p = 2, \text{ with } V_4 = \langle s, t \mid s^2, t^2, (st)^2 \rangle \end{cases}
$$

and the corner voltage assignment v as follows: $v(x)$ only has a non–trivial value in the component $\langle a, b \rangle.f_x$, where $x \in \langle a, b \rangle.f_x$. For the elements of this orbit, its value is defined as

$$
\begin{aligned}
f_x &\mapsto 1 & ab.f_x &\mapsto 1 & abab.f_x &\mapsto p - 2 \\
b.f_x &\mapsto p - 1 & bab.f_x &\mapsto p - 1 & babab.f_x &\mapsto 2
\end{aligned}
$$

for odd p and as

$$
\begin{aligned}
f_x &\mapsto s & ab.f_x &\mapsto t & abab.f_x &\mapsto st \\
b.f_x &\mapsto s & bab.f_x &\mapsto t & babab.f_x &\mapsto st
\end{aligned}
$$

for $p = 2$. By Remark 9.5.10, the lift via v produces a surface $(D, \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2)$ (after restriction to one orbit of $\langle \hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2 \rangle$ on $H_d/\{1\} \times B$).

We compute the cycle lengths of $\hat{\sigma}_1\hat{\sigma}_2$ (umbrellas) and $\hat{\sigma}_1\hat{\sigma}_0\hat{\sigma}_2$ (geodesics). Since the argument for them is similar, we only give the case for $\hat{\sigma}_1\hat{\sigma}_2$.

Let $\hat{f} = (x, g) \in D \subseteq H_d/\{1\} \times B$. If $\langle a, b \rangle.(bc)^k x = \langle a, b \rangle.x$, we conclude $(bc)^k \in \langle a, b \rangle$. By assumption, this is only possible if k is a multiple of d. In particular, $(\hat{\sigma}_1\hat{\sigma}_2)^d.\hat{f} = (x, wg)$, with $w \in B$ such that w has a non–trivial entry at each $(\hat{\sigma}_1\hat{\sigma}_2)^k.x$ (for $0 \leq k < d$). By the previous analysis, these positions are all distinct. Since all non–trivial elements in $\mathbb{Z}/p\mathbb{Z}$ have order p and all non–trivial elements in V_4 have order 2, the order of $\hat{\sigma}_1\hat{\sigma}_2$ is dp.

To show the additional claim, observe that T_{dp} acts transitively on D (Remark 4.3.10). The element $(bac)^{dp} \in T_{dp}$ acts trivially, thus $H_{dp} = T_{dp}/\langle\langle (bac)^{dp} \rangle\rangle$ also acts transitively on $\mathcal{G}$. But the *bac*–orbits of H_d have maximal length d, so $H_d \neq H_{dp}$. □

Proposition 9.5.15. *H_{p^n} is uncollapsed for $p > 3$ prime and $n > 1$.*

Proof. We show the claim by induction. By Corollary 9.5.5, H_{25} and H_{49} are uncollapsed, together with all H_p with $p > 10$ prime. Also, all of them are infinite by Remark 9.4.1 and satisfy the assumption of proposition 9.5.14 by Lemma 9.5.12. □

Proposition 9.5.16. *H_{4p} is uncollapsed for all odd primes $p > 3$.*

Proof. By Lemma 9.5.4, we only have to consider H_4 and H_{2p}. From Remark 9.4.1, clearly $H_4 \neq H_{4p}$. Further, H_{2p} is infinite ($p > 3$), uncollapsed (Lemma 9.5.6) and fulfils the assumption of Proposition 9.5.14 (Lemma 9.5.13). □

9.5.2 Uncollapsed induction

In Subsection 9.5.1, we showed that several classes of geodesic triangle groups are uncollapsed. In this subsection, we combine these results to show that almost all geodesic triangle groups are uncollapsed. The central observation is the following lemma:

Lemma 9.5.17. *Let $d = k \cdot p$ for a prime p such that there is a $z \mid k$ with $p \nmid z$. Then, $H_d = H_k$ implies $H_{\frac{d}{z}} = H_{\frac{k}{z}}$.*

Proof. Clearly, $\frac{k}{z} \mid \gcd(k, \frac{kp}{z})$. If $p^n \mid k$, then $p^{n+1} \mid \frac{kp}{z}$, since $p \nmid z$. Therefore, $\frac{k}{z} = \gcd(k, \frac{kp}{z})$ and we have $H_{\frac{d}{z}} = H_d/\langle\langle (bc)^{\frac{d}{z}}, (bac)^{\frac{d}{z}} \rangle\rangle = H_k/\langle\langle (bc)^{\frac{d}{z}}, (bac)^{\frac{d}{z}} \rangle\rangle = H_{\frac{k}{z}}$. □

Theorem 9.5.18. *H_d is uncollapsed for all $d \geq 5$ with $d \neq 7$.*

Proof. Assume H_d is a counterexample. By Lemma 9.5.4, there is a prime p with $d = p^{n+1} z$ with $n \geq 0$ and $p \nmid z$, such that $H_{p^n z} = H_d$.

By Lemma 9.5.17, this implies $H_{p^n} = H_{p^{n+1}}$. We distinguish several cases:

- $p = 2$: By Theorem 9.5.7, this is only possible if $n \in \{0, 1\}$. Corollary 9.5.5 restricts this further to $n = 1$.

- $p = 3$: By Theorem 9.5.7, this is only possible for $n = 0$.
- $p = 7$: By Proposition 9.5.15, this is only possible for $n = 0$.
- In all other cases, the combination of Theorem 9.5.7, Proposition 9.5.15 and Corollary 9.5.5 makes this situation impossible.

For the three remaining cases, we apply Lemma 9.5.17 to $d = p^{n+1}z$ in a different way: Let q be a prime dividing z. Then we can use Lemma 9.5.17 to divide by $\frac{z}{q}$. This gives the three cases $H_{4q} = H_{2q}$, $H_{3q} = H_q$, and $H_{7q} = H_q$.

The first one is impossible by Proposition 9.5.16 (for $p > 3$) and Corollary 9.5.5 (for $p = 3$). If $q \in \{3, 5, 7\}$, the impossibility of the other cases follow from Corollary 9.5.5. Otherwise, H_q satisfies the assumptions of Proposition 9.5.14 (infinite by Remark 9.4.1, so Lemma 9.5.12 holds). □

9.6 Classification

In this section, we complete the proof of the main theorem. Afterwards, we give a complete classification of all geodesic self–dual degree–d–surfaces for $d < 10$. Several of these surfaces are also available in the GAP–package `SimplicialSurfaces` ([13]). In these cases, we will also give the command to generate this particular surface.

Recall Definition 9.3.2 of the geodesic triangle group H_d and Definition 9.3.4 of the geodesic automorphism $\# : H_d \to H_d$.

Theorem 9.6.1. *Let $V \leq H_d$ with $d \geq 5$ and $d \neq 7$. Then $(H_d/V, a, b, c)$ is a geodesic self–dual degree–d–surface if and only if*

- *$g^{-1}Vg \cap X = \{1\}$ for $X \in \{\langle a\rangle, \langle c\rangle, \langle ab\rangle, \langle ac\rangle, \langle bc\rangle\}$ and all $g \in H_d$.*
- *$V^{\#}$ is conjugate to V.*

Furthermore, all geodesic self–dual degree–d–surfaces have this form.

Proof. This follows from Corollary 9.4.5 and Theorem 9.5.18, by using the reformulation from Lemma 9.5.2 and Definition 9.5.3. □

For $d \in \{5, 6, 8, 9\}$, the group H_d is finite, so we can use GAP ([33]) to compute all geodesic self–dual degree–d–surfaces.

Example 9.6.2. *For $d = 5$, there is only one geodesic self–dual surface, since only the trivial subgroup $\{1\}$ satisfies Theorem 9.6.1. This defines the projective plane on 10 triangles (6 vertices and 15 edges), shown in figure 9.4. The associated command in the `SimplicialSurfaces`–package is `AllGeodesicSelfDualSurfaces(10)[1]`.*

Example 9.6.3. *For $d = 6$, there are exactly two geodesic self–dual surfaces, since there are exactly two surface subgroups satisfying Theorem 9.6.1 (both are tori):*

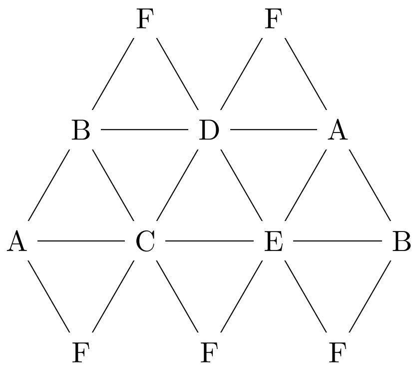


Figure 9.4: Geodesic self–dual degree–5–surface

1. *The trivial subgroup* $\{1\}$*, defining a surface with 18 faces (9 vertices and 27 edges). Its command is* `AllGeodesicSelfDualSurfaces(18)[1]`.

2. *A normal subgroup of size 3, defining a surface with 6 faces (3 vertices and 9 edges). Its command is* `AllGeodesicSelfDualSurfaces(6)[1]`.

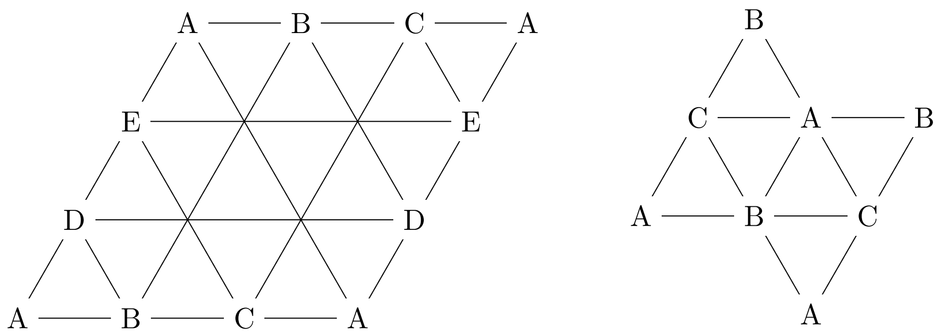


Figure 9.5: Geodesic self–dual degree–6–surfaces

Example 9.6.4. *For* $d = 8$*, there are exactly four geodesic self–dual surfaces, defined by the four surface subgroups satisfying Theorem 9.6.1 (up to conjugation):*

1. *The trivial subgroup* $\{1\}$*, defining an orientable surface with 42 vertices, 168 edges, and 112 faces (genus 8).*

   Its command is `AllGeodesicSelfDualSurfaces(112)[1]`.

2. *A normal subgroup of size 2, defining a non–orientable surface with 21 vertices, 84 edges, and 56 faces (genus 8).*

   Its command is `AllGeodesicSelfDualSurfaces(56)[1]`.

3. *A subgroup of size 7 and index 96, defining an orientable surface with 6 vertices, 24 edges, and 16 faces (genus 2).*
4. *A subgroup of size 14 and index 48, defining a non–orientable surface with 3 vertices, 12 edges, and 8 faces (genus 2).*

Example 9.6.5. *For $d = 9$, there are exactly three geodesic self–dual surfaces, defined by the three surface subgroups satisfying Theorem 9.6.1 (up to conjugation):*

1. *The trivial subgroup $\{1\}$, defining a non–orientable surface with 190 vertices, 855 edges, and 570 faces (genus 96).*

   Its command is `AllGeodesicSelfDualSurfaces(570)[1]`.

2. *A group of size 5 with index 684, defining a non–orientable surface with 38 vertices, 171 edges, and 114 faces (genus 20).*
3. *A group of size 19 with index 180, defining a non–orientable surface with 10 vertices, 45 edges, and 30 faces (genus 6).*

At this point, we have classified all geodesic self–dual degree–d–surfaces for $d < 10$. For $d \geq 10$, the situation is unclear: We conjecture that there are infinitely many geodesic self–dual surfaces for each $d \geq 10$. This is based on the observation that our calculations reached their computational limits before stopping to construct further examples.

Unfortunately, it is still unclear whether the geodesic surface subgroups of H_d for $d \geq 10$ can be characterised in a fashion that is more amenable to analysis.

10 `GAP`–package `SimplicialSurfaces`

The chapters so far have been concerned with the theory of combinatorial surfaces. In contrast, this chapter has a more practical flavour. It presents the `GAP`–package ([33]) `SimplicialSurfaces` ([13]), which performs computations with combinatorial surfaces. It is co–authored with Alice Niemeyer, but the vast majority of design and implementation is part of this thesis. The aims of this package are threefold:

- Computing with the package is faster than computing by hand, which allows the user to focus more on mathematical structure and less on menial computation.
- The package contains a wide variety of surfaces which can be used to test theories against.
- The package makes it easier to implement custom code dealing with combinatorial surfaces and can be easily extended by someone interested in research about combinatorial surfaces.

The functionality of the package is quite extensive, so this chapter does not detail every single method (for that, we refer to the package documentation in [13]). Instead, we focus on the general themes of the package.

In Section 10.1, we explore which concepts are implemented in the package and how its notation relates that of this thesis. Section 10.2 presents several convenient features of the package that are often useful in practice, including isomorphism testing, surface drawing, and the surface library. Section 10.3 explores the flexible uses allowed by the package. This is most important to those researchers whose research questions are not covered by the pre–defined methods.

10.1 Notation and Usage

This section explains the notation for combinatorial surfaces that is used in the `GAP`–package `SimplicialSurfaces` ([13]) and how it relates to the notation used in this thesis.

Subsection 10.1.1 covers the methods to construct combinatorial surfaces and to access their incidence structure. In Subsection 10.1.2, homomorphisms between combinatorial surfaces are discussed, and Subsection 10.1.3 contains the different path–concepts. Some basic properties of combinatorial surfaces are touched upon in Subsection 10.1.4 and the final Subsection 10.1.5 deals with edge–colourings.

10.1.1 Constructing complexes and surfaces

In this thesis, we developed three formalisms to capture the concept of combinatorial surfaces: twisted polygonal complexes (Section 2.4), polygonal complexes (Section 2.5), and Dress–surfaces (Section 2.6). The package `SimplicialSurfaces` primarily relies on the concept of polygonal complexes and does not currently support twisted polygonal complexes in an extended capacity. Furthermore, it contains methods to convert the description of Dress–surfaces into polygonal surfaces.

A polygonal complex (compare Definition 2.5.2) is a quintuple (V, E, F, η, φ), where V, E, and F are sets, and $\eta : E \to \mathrm{Pot}_2(V)$ and $\varphi : F \to \mathrm{Pot}(E)$ are maps satisfying certain properties. In the package `SimplicialSurfaces`, they are represented as follows:

- The sets are represented by `GAP`–sets of positive integers.
- The map η is represented by a `GAP`–list L, where the i–th component is the `GAP`–set formed from the elements in $\eta(i)$ (we identify the elements of E with the entry positions of L). This list can be accessed by the method `VerticesOfEdges`.
- The map φ is represented in the same way as η. This list can be accessed by the method `EdgesOfFaces`.

Example 10.1.1. *Consider the polygonal complex* (V, E, F, η, φ), *with*

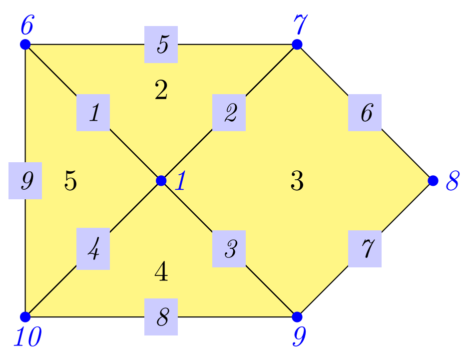
$$V = \{1, 6, 7, 8, 9, 10\}, \qquad E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \qquad F = \{2, 3, 4, 5\},$$

and

$$\eta : E \to \mathrm{Pot}_2(V) \qquad e \mapsto \begin{cases} \{1, 5 + e\} & 1 \le e \le 2 \\ \{1, 6 + e\} & 3 \le e \le 4 \\ \{e + 1, e + 2\} & 5 \le e \le 8 \\ \{6, 10\} & e = 9, \end{cases}$$

$$\varphi : F \to \mathrm{Pot}(E) \qquad f \mapsto \begin{cases} \{1, 2, 5\} & f = 2 \\ \{2, 3, 6, 7\} & f = 3 \\ \{3, 4, 8\} & f = 4 \\ \{1, 4, 9\} & f = 5, \end{cases}$$

illustrated by:



In this case, the GAP–list `VerticesOfEdges` would be

```
gap> VofE := [ [1,6], [1,7], [1,9], [1,10],
>                    [6,7], [7,8], [8,9], [9,10], [6,10] ];;
```

(GAP–sets are stored as GAP–lists) and `EdgesOfFaces` would be

```
gap> EofF := [ , [1,2,5], [2,3,6,7], [3,4,8], [1,4,9] ];;
```

With these two lists, the polygonal complex (V, E, F, η, φ) can be constructed with the method `PolygonalComplexByDownwardIncidence`:

```
gap> PolygonalComplexByDownwardIncidence( VofE, EofF );
polygonal surface (6 vertices, 9 edges, and 4 faces)
```

Example 10.1.1 shows the method `PolygonalComplexByDownwardIncidence`. There is also a method `PolygonalComplexByUpwardIncidence` that uses the maps

$$\begin{aligned} \bar{\eta} &: V \to \mathrm{Pot}(E) & v &\mapsto \{e \in E \mid v \in \eta(e)\} \\ \bar{\varphi} &: E \to \mathrm{Pot}(F) & e &\mapsto \{f \in F \mid e \in \varphi(f)\}. \end{aligned}$$

Example 10.1.2. *Let (V, E, F, η, φ) be the polygonal complex from Example 10.1.1. Let $\bar{\eta} : V \to \mathrm{Pot}(E), \quad v \mapsto \{e \in E \mid v \in \eta(e)\}$, then*

$$\begin{aligned} \bar{\eta}(1) &= \{1,2,3,4\} & \bar{\eta}(6) &= \{1,5,9\} & \bar{\eta}(7) &= \{2,5,6\} \\ \bar{\eta}(8) &= \{6,7\} & \bar{\eta}(9) &= \{3,7,8\} & \bar{\eta}(10) &= \{4,8,9\}, \end{aligned}$$

encoded as the GAP–list

```
gap> EofV := [ [1,2,3,4], , , , , [1,5,9],
>               [2,5,6], [6,7], [3,7,8], [4,8,9] ];;
```

Let $\bar{\varphi} : E \to \mathrm{Pot}(F), \quad e \mapsto \{f \in F \mid e \in \varphi(f)\}$, then

$$\bar{\varphi}(e) = \begin{cases} \{2,5\} & e = 1 \\ \{e, e+1\} & 2 \le e \le 4 \\ \{e-3\} & 5 \le e \le 6 \\ \{e-4\} & 7 \le e \le 9, \end{cases}$$

encoded as the GAP–list

```
gap> FofE := [ [2,5], [2,3], [3,4], [4,5],
>                    [2], [3], [3], [4], [5] ];;
```

Then, (V, E, F, η, φ) can be constructed as follows:

```
gap> PolygonalComplexByUpwardIncidence( EofV, FofE );
polygonal surface (6 vertices, 9 edges, and 4 faces)
```

If we only want to construct vertex–faithful polygonal complexes, the map $\eta \between \varphi$ is sufficient (like in Lemma 2.7.5), with the method `PolygonalComplexByVerticesInFaces`.

Constructing the polygonal surface corresponding to the Dress–surface $(C, \mathtt{a}, \mathtt{b}, \mathtt{c})$ is a bit more involved. If the involutions are given as `GAP`–permutations, the following code constructs the polygonal surface:

```
gap> tame := AllTameColouredSurfaces(a,b,c,[1,1,1])[1];;
gap> surf := IsomorphicFlagSurface(tame);;
```

If the Dress–surface cannot be represented by a polygonal surface, the method returns `false`.

10.1.2 Homomorphisms

The package `SimplicialSurfaces` supports polygonal morphisms. By Definition 2.5.6, a polygonal morphisms is a triple of maps (μ_V, μ_E, μ_F) satisfying certain consistency criteria.

Each of these maps can be represented by a `GAP`–list. Consider the map μ_V. If the vertex sets only contain positive integers (by Subsection 10.1.1, this is fulfilled for the polygonal complexes in the package), we can represent μ_V as a `GAP`–list, where the position v contains the entry $\mu_V(v)$. In the same way, the maps μ_E and μ_F can be encoded.

For example, the map

$$\mu_V : \{1,2,4,6\} \to \{1,3,5\} \qquad\qquad v \mapsto \begin{cases} 1 & v \in \{1,4\} \\ 3 & v = 2 \\ 5 & v = 6 \end{cases}$$

would be represented by the `GAP`–list $[1,3,,1,,5]$.

Currently, there are two ways to construct polygonal morphisms:

1. The method `PolygonalMorphismByLists` constructs a polygonal morphism between general polygonal complexes from three `GAP`–lists that correspond to the maps μ_V, μ_E, and μ_F.

2. The method `PolygonalMorphismByVertexImages` constructs a polygonal morphism between vertex–faithful polygonal complexes from the `GAP`–list corresponding to the map μ_V.

10.1.3 Paths

The package `SimplicialSurfaces` implements vertex–edge–paths (Definition 5.2.10) and edge–face–paths (Definition 2.5.17). Both can be constructed in different ways:

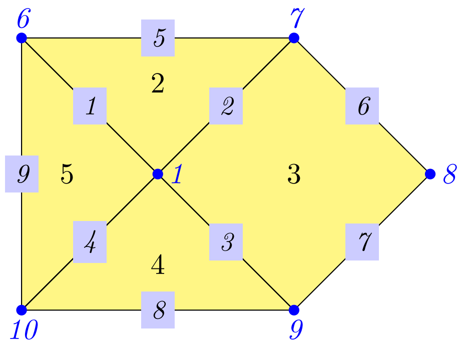
- The generic way to construct them is by giving a `GAP`–list whose entries alternate between vertices and edges (or edges and faces). The methods are `VertexEdgePath` and `EdgeFacePath`.

- For vertex–edge–paths, we can also give a `GAP`–list with the sequence of vertices (`VertexEdgePathByVertices`) or edges (`VertexEdgePathByEdges`). However, this might not determine a vertex–edge–path uniquely. Then, the methods return one of them

- Similarly, for edge–face–paths, we can give a `GAP`–list with the sequence of edges (or faces). The method is `EdgeFacePathByEdges` (or `EdgeFacePathByFaces`).

To work with these paths one needs access to the vertices, edges, and faces contained in them. These can be accessed by the methods `VerticesAsList` and `EdgesAsList` (for vertex–edge–paths), as well as `EdgesAsList` and `FacesAsList` (for edge–face–paths).

Example 10.1.3. *Let (V, E, F, η, φ) be the polygonal complex from Example 10.1.1, illustrated by*



and defined by

```
gap> complex := PolygonalComplexByDownwardIncidence(
>       [[1,6],[1,7],[1,9],[1,10],[6,7],[7,8],[8,9],[9,10],[6,10]],
>       [,[1,2,5],[2,3,6,7],[3,4,8],[1,4,9]]);;
```

To construct the vertex–edge–path that moves along the boundary clockwise, we have several options:

```
gap> genPath := VertexEdgePath(complex, [6,5,7,6,8,7,9,8,10,9,6]);
( v6, E5, v7, E6, v8, E7, v9, E8, v10, E9, v6 )
gap> vtxPath := VertexEdgePathByVertices(complex, [6,7,8,9,10,6]);
( v6, E5, v7, E6, v8, E7, v9, E8, v10, E9, v6 )
gap> edgePath := VertexEdgePathByEdges(complex, [5,6,7,8,9]);
( v6, E5, v7, E6, v8, E7, v9, E8, v10, E9, v6 )
```

We can access the included vertices and edges in the correct order:

```
gap> VerticesAsList(genPath);
[ 6, 7, 8, 9, 10, 6 ]
gap> EdgesAsList(genPath);
[ 5, 6, 7, 8, 9 ]
```

Some of these paths are distinguished:

- A vertex–edge–path where all vertices and edges are incident to the same face is called *perimeter path*. They can be constructed by `PerimeterPathsOfFaces`.
- It is possible to recognise whether an edge–face–path is an umbrella–path (compare Definition 2.5.20). The umbrella partition from Lemma 2.5.24 can be computed with `UmbrellaPathPartitionOfVertices`.
- In Chapter 9, the concept of *geodesic paths* is introduced. These are also included in the package.

Example 10.1.4. *Consider the polygonal complex (V, E, F, η, φ) from Example 10.1.1 and Example 10.1.3, defined by:*

```
gap> complex := PolygonalComplexByDownwardIncidence(
>       [[1,6],[1,7],[1,9],[1,10],[6,7],[7,8],[8,9],[9,10],[6,10]],
>       [,[1,2,5],[2,3,6,7],[3,4,8],[1,4,9]]);;
```

We are interested in perimeter and umbrella paths.

```
gap> perims := PerimeterPathsOfFaces(complex);;
gap> perims[2];
( v1, E1, v6, E5, v7, E2, v1 )
gap> perims[3];
( v1, E2, v7, E6, v8, E7, v9, E3, v1 )
```

Next, we take a look at the umbrella paths.

```
gap> umbs := UmbrellaPathsOfVertices(complex);;
gap> umbs[1];
( e1, F2, e2, F3, e3, F4, e4, F5, e1 )
gap> umbs[6];
| e5, F2, e1, F5, e9 |
```

Note that the representation of the edge–face–paths already tells us whether it is closed or not (compare Definition 2.5.17).

10.1.4 Basic properties

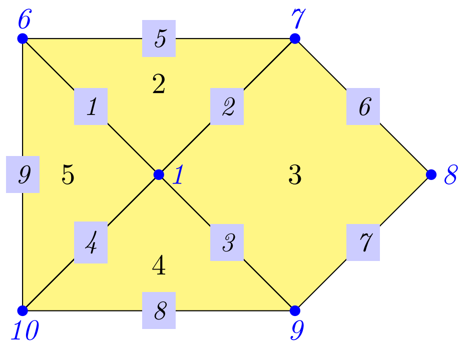
So far, this section explained definitions within the package `SimplicialSurfaces`. This subsection collects some of the “basic” properties that are easy to calculate with the package.

- We can compute the different vertex types from Definition 2.5.25 with the methods `InnerVertices`, `BoundaryVertices`, `RamifiedVertices`, and `ChaoticVertices`.
- We can compute the different edge types from Definition 2.5.15 with the methods `InnerEdges`, `BoundaryEdges`, `RamifiedEdges`, and `ChaoticEdges`.

- The concepts *connectivity* and *strong connectivity* (from Section 5.2) can be computed with the methods `IsConnected` and `IsStronglyConnected`.

- Section 5.3 introduced the concepts of *orientation* and *dual orientation*. Currently, only *orientation* is implemented and can be computed with the method `IsOrientable`.

Example 10.1.5. *Let* (V, E, F, η, φ) *be the polygonal complex from Example 10.1.1, illustrated by*



and defined by

```
gap> complex := PolygonalComplexByDownwardIncidence(
>       [[1,6],[1,7],[1,9],[1,10],[6,7],[7,8],[8,9],[9,10],[6,10]],
>       [,[1,2,5],[2,3,6,7],[3,4,8],[1,4,9]]);;
```

We compute the types of its vertices and edges.

```
gap> InnerVertices(complex);
[ 1 ]
gap> BoundaryVertices(complex);
[ 6, 7, 8, 9, 10 ]
gap> InnerEdges(complex);
[ 1, 2, 3, 4 ]
gap> BoundaryEdges(complex);
[ 5, 6, 7, 8, 9 ]
```

Furthermore, it is orientable and strongly connected.

```
gap> IsStronglyConnected(complex);
true
gap> IsOrientable(complex);
true
```

In fact, the package can also provide the local orientation map (compare Definition 5.3.1). It stores them as perimeter paths (compare Subsection 10.1.3). The vertex permutation can be accessed by `VerticesAsPerm`.

```
gap> orient := Orientation(complex);;
gap> VerticesAsPerm(orient[2]);
(1,6,7)
gap> VerticesAsPerm(orient[3]);
(1,7,8,9)
gap> List(orient, VerticesAsPerm);
[ , (1,6,7), (1,7,8,9), (1,9,10), (1,10,6) ]
```

10.1.5 Edge–colourings

The package `SimplicialSurfaces` supports edge–colourings (Subsection 3.3.1). They are formed with the method `EdgeColouredPolygonalComplex` from a polygonal complex and a map from edges to colours (positive integers), which we encode as a `GAP`–list, similar to the encoding in Subsection 10.1.2. We can access this `GAP`–list by `ColoursOfEdges`.

Starting from general edge–colourings, the package also deals with some restricted versions. Here, we focus on Grünbaum colourings (Definition 3.3.3), called *wild colourings* in the package. We can check whether a given edge–coloured triangular surface has a Grünbaum colouring with `IsWildColouredSurface`. We can also compute the local symmetry (Definition 3.3.4) with the method `LocalSymmetryOfEdges`.

Each colour of a Grünbaum colouring can be represented by an involution. We can access these involutions by the method `ColourInvolutions`. We can go in the converse direction as well: Given three involutions, we can construct Grünbaum colourings with specified local symmetry, with `AllWildColouredSurfaces`. This allows us to input Dress–surfaces into the package (for details, compare the end of Subsection 10.1.1).

10.2 Frequently used features

Section 10.1 primarily dealt with the notation and basic functionality of the `GAP`–package `SimplicialSurfaces` ([13]). In this section, we explore several features that are quite useful in practice.

- Subsection 10.2.1 presents the surface library of the package. It allows the user to directly work with several surface examples and to test theories about them.
- Subsection 10.2.2 covers both isomorphism testing and automorphism computation. They rely on `Nauty` by McKay and Piperno ([52]).
- Subsection 10.2.3 explains how to draw nets of surfaces into the plane, according to certain specifications.

10.2.1 Surface Library

In many cases, a well–chosen counterexample inspires a lot of progress. Unfortunately, finding such an example is often difficult. The surface library within the package

SimplicialSurfaces ameliorates this problem by defining several lists of surfaces that can be used to test theories against.

These lists can be used in two mayor ways:

1. As a list of surfaces that can be filtered according to certain criteria. The method is AllPolygonalComplexes.

2. As a classification of surfaces with certain properties.

Most of the surfaces in the package are part of a classification. Currently, there are three classifications:

- All platonic solids (AllPlatonicSolids).

- All simplicial spheres without three waists (i. e. vertex–edge–paths with three edges that are not all incident to the same face) and at most 28 faces. They can be called with AllSimplicialSpheres.

  This classification was carried out in [56] and [65] for simplicial spheres. For the formalism of planar graphs, the methods for the classification already appeared in [19].

- All geodesic self–dual surfaces with $d < 10$ (classified in Chapter 9), as long as they are polygonal complexes. The method is AllGeodesicSelfDualSurfaces.

10.2.2 Isomorphisms and automorphisms

It is a common question whether two given surfaces are "the same". In a software package, the notion of equality is usually too strict, since the labels of vertices, edges, and faces might differ depending on the methods that produced them. Thus, we care about *isomorphisms*.

It is possible to encode the incidence structure of a polygonal complex as a graph (consult the package documentation in [13] for details). Then, we can use Nauty ([52]) to check for isomorphism with IsIsomorphic. This description also allows us to compute the automorphism group of a polygonal complex efficiently (with AutomorphismGroup).

The package SimplicialSurfaces also allows the user to reference the action of the automorphism group on vertices, edges, and faces separately, by methods such as AutomorphismGroupOnVertices.

10.2.3 Surface Drawing

A big restriction of GAP is its lack of visualisation. It is a console program, which makes manipulation of combinatorial surfaces (that are visual in many ways) complicated. To work around this limitation, the package SimplicialSurfaces contains a method to construct nets of polygonal surfaces (with chosen edge lengths).

These nets can be customised according to certain criteria (like colours and labels) and are outputted as TEX–file in the Ti*k*Z–format ([68]). A comprehensive introduction into the capabilities of the drawing method can be found in the package documentation ([13]).

As an example, we construct a net of the octahedron.

```
gap> oct := Octahedron();
simplicial surface (6 vertices, 12 edges, and 8 faces)
gap> pr := DrawSurfaceToTikz( oct, "Octahedron_example" );;
```

This code writes a file *Octahedron_example.tex* that contains the image in Figure 10.1.

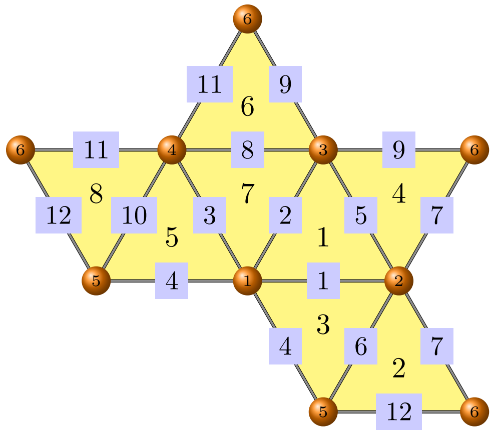


Figure 10.1: Net of an octahedron, computed in the package

To exemplify the possible customisations, we

1. Change the size of the image.
2. Change the colour of the vertices.
3. Change the labels of the faces.
4. Change a few edge lengths.

The code of this example is explained in detail in the package documentation ([13]).

```
gap> pr.scale := 3;;
gap> pr.vertexColours := "green";;
gap> pr.faceLabels := ["I","II","III","IV","V","VI","VII","VIII"];;
gap> pr.edgeLengths := [1,1,1,1,1.5,1.5,1,1.5,1,1.5,1,1];;
gap> Unbind( pr.angles );
gap> DrawSurfaceToTikz( oct, "Octahedron_reshaped", pr );;
```

The result of these changes is shown in Figure 10.2. The package recomputed the net completely to avoid intersections.

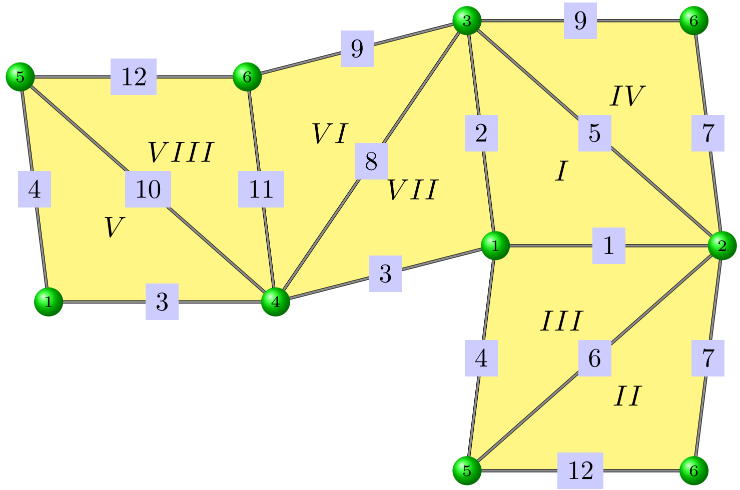


Figure 10.2: Modified net of an octahedron, computed in the package

10.3 Flexible usage

In Section 10.1, we explored basic notation and elementary usage of the `GAP`–package `SimplicialSurfaces` ([13]). In Section 10.2, we explored several helpful features that can be used immediately.

All of these have a fundamentally *static* nature: There is a clearly defined use–case to which a potential user has to adapt. In many applications, more flexibility is needed and the package aims to accommodate this sort of application as well. However, this requires a trade–off: To allow greater flexibility, the individual methods become less impressive in their own right (since it falls to the user to combine them into something greater).

Currently, there are two main parts of the package that are built with this kind of flexibility in mind:

- Subsection 10.3.1 presents methods to navigate in a polygonal complex. This includes movement to adjacent structures as well as localising specific structures that one is interested in.

- Subsection 10.3.2 presents methods to modify surfaces. These methods can be used as a toolbox to construct arbitrary modifications.

10.3.1 Navigation in a surface

In this subsection, we present methods to obtain detailed knowledge about a polygonal complex. We work with two main scenarios. In the first one, we start with a subconfiguration of the polygonal complex and explore its neighbourhood. In the second one, we find subconfigurations with certain properties.

There are three ways to explore the neighbourhood of a subconfiguration:

- We can move along edges. To do so, we need to check adjacency of vertices (`IsVerticesAdjacent`), and, given a vertex and incident edge, find the other vertex incident to the edge (`OtherVertexOfEdge`).
- We can move within faces. In general, this can be done along the boundary (`OtherEdgeOfVertexInFace`). If the face is a triangle, we can also go from a vertex to the "opposite" edge (`OppositeEdgeOfVertexInTriangle`) and vice versa (`OppositeVertexOfEdgeInTriangle`).
- We can move between adjacent faces. For that, we need to check adjacency of faces (`IsFacesAdjacent`), and to find adjacent faces (`NeighbourFacesByEdge`).

These methods are basic but they allow rather flexible movement within a polygonal complex. In contrast, the methods to localise subconfigurations usually require a bit more setup. They can find:

- All adjacent vertices fulfilling certain properties (`EdgesWithVertexProperties` and `AdjacentVerticesWithProperties`).
- All faces whose vertices fulfil certain properties (`FacesWithVertexProperties`).
- All faces whose edges fulfil certain properties (`FacesWithEdgeProperties`).

All of these descriptions seem kind of vague since they contain the phrase "certain properties". This vagueness is part of the design. While it is possible to use predefined properties (e. g. the degree of a vertex, or whether an edge is ramified), the strength of these methods becomes apparent when called with custom tailored properties

In `GAP`, any user can define their own `GAP`–functions. The "properties" of these methods are just `GAP`–functions satisfying a certain form. Thus, any property that can be computed from a polygonal complex and a vertex (or an edge), can be used to localise subconfigurations.

10.3.2 Modifying surfaces

In some cases, it is sufficient to work with a set of pre–defined surfaces. In many others, however, modifying surfaces is a very important part of the work. Unfortunately, there are too many different modifications to implement them all. Thus, it is necessary that a user defines these modifications by themselves.

A naive approach is to modify the incidence structure of a polygonal complex directly. While this works (and results in fastest computations), it is usually quite tedious and error–prone. This makes it infeasible for situations in which one wants to quickly test a theory without committing too much time to it.

The solution of the package `SimplicialSurfaces` is defining a toolbox for surface modifications. This is a small set of methods from which almost all other modifications can be constructed. More concretely, these building blocks are:

- Split the surface along a vertex–edge–path with `SplitVertexEdgePath`. As special cases, we can also split along an edge (`SplitEdge`) or at a vertex (`SplitVertex`). For the details of splitting, we refer to the manual of the package ([13]).
- Remove faces (`RemoveFaces`) or construct a subsurface (`SubsurfaceByFaces`).
- Combine two surfaces disjointly with `DisjointUnion`. This usually requires a relabelling of one of them.
- Join one (or two) surfaces along a vertex–edge–path with `JoinVertexEdgePaths` (inverse to the splitting operation). As a special case, the method `JoinBoundaries` joins two surfaces along their boundaries. For more details, we refer to the package documentation ([13]).

In the remainder of this chapter, we construct the vertex–splitting operation from Definition 8.1.1 with this toolbox. We follow the intuitive description from the start of Section 8.1:

1. Choose two edges that are incident to the same vertex.
2. Cut along these edges. This leaves a hole with four boundary edges.
3. Insert two triangles into the hole.

We write a `GAP`–function that takes a surface `surf` and two edges $e1$ and $e2$.

```
gap> VertexSplit := function( surf, e1, e2 )
>       ...
> end;;
```

It remains to fill in the body of this method. We do so in several steps.

1. **Input validation:**

   Depending on the context, we want to check whether the input is valid. Checking whether `surf` is a surface is easy.

   ```
   gap> if not IsPolygonalSurface(surf) then
   >        return fail;
   >    fi;
   ```

   There should be exactly one vertex that is incident to both edges. Additionally, this should be an inner vertex.

   ```
   gap> incVerts := Intersection( VerticesOfEdge(surf,e1),
   >        VerticesOfEdge(surf,e2));
   gap> if Length(incVerts) <> 1 then
   >        return fail;
   >    fi;
   gap> if not IsInnerVertex(surf, incVerts[1]) then
   >        return fail;
   >    fi;
   ```

2. **Cut along the edges:**

   Since there is exactly one vertex incident to both edges, there is exactly one vertex–edge–path with two edges that contains both of them.

   ```
   gap> path := VertexEdgePathByEdges( surf, [e1,e2] );
   ```

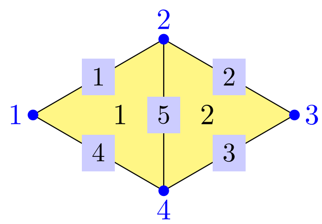
   Next, we cut along this path.

   ```
   gap> cut := SplitEdgePath( surf, path );
   gap> pathA := cut[2][1][1];
   gap> pathB := cut[2][2][1];
   ```

   Here, `pathA` and `pathB` are the two vertex–edge–paths that result from the split.

3. **Construct new triangles:**

   We want to fill in two triangles into the resulting hole. “Filling in” corresponds to joining two surfaces. Thus, we have to construct a polygonal surface from two triangles, corresponding to the following illustration:


   ```
   gap> fill := SimplicialSurfaceByDownwardIncidence(
   >        [[1,2],[2,3],[3,4],[1,4],[2,4]],[[1,4,5],[2,3,5]]);
   ```

4. **Combine the surfaces:**

   We start by forming the disjoint union of the two surfaces.

   ```
   gap> res := DisjointUnion( cut[1], fill );
   gap> union := res[1];
   gap> shift := res[2];
   ```

   Then, we combine `pathA` with the vertex–edge–path along the (former) edges 1 and 2 of `fill`.

   ```
   gap> fillA := VertexEdgePathByEdges(union, [1+shift,2+shift]);
   gap> holeA := VertexEdgePathByEdges(union, EdgesAsList(pathA));
   gap> join := JoinVertexEdgePaths(union, holeA, fillA)[1];
   ```

   Next, we combine `pathB` with the vertex–edge–path along the (former) edges 4 and 3 of `fill`.

   ```
   gap> fillB := VertexEdgePathByEdges(join, [4+shift,3+shift]);
   gap> holeB := VertexEdgePathByEdges(join, EdgesAsList(pathB));
   gap> final := JoinVertexEdgePaths(join, holeB, fillB)[1];
   ```

In combination, we obtain the full method.

```
gap> VertexSplit := function( surf, e1, e2 )
>       local invVerts, path, cut, pathA, pathB, fill, res,
>           union, shift, fillA, holeA, join, fillB, holeB, final;
>       # Input validation
>       if not IsPolygonalSurface(surf) then
>           return fail;
>       fi;
>       incVerts := Intersection( VerticesOfEdge(surf,e1),
>           VerticesOfEdge(surf,e2));
>       if Length(incVerts) <> 1 then
>           return fail;
>       fi;
>       if not IsInnerVertex(surf, incVerts[1]) then
>           return fail;
>       fi;
>       # Cut along the edges
>       path := VertexEdgePathByEdges( surf, [e1,e2] );
>       cut := SplitEdgePath( surf, path );
>       pathA := cut[2][1][1];
>       pathB := cut[2][2][1];
>       # Construct new triangles
>       fill := SimplicialSurfaceByDownwardIncidence(
>           [[1,2],[2,3],[3,4],[1,4],[2,4]],[[1,4,5],[2,3,5]]);
>       # Combine the surfaces
>       res := DisjointUnion( cut[1], fill );
>       union := res[1];
>       shift := res[2];
>       fillA := VertexEdgePathByEdges(union, [1+shift,2+shift]);
>       holeA := VertexEdgePathByEdges(union, EdgesAsList(pathA));
>       join := JoinVertexEdgePaths(union, holeA, fillA)[1];
>       fillB := VertexEdgePathByEdges(join, [4+shift,3+shift]);
>       holeB := VertexEdgePathByEdges(join, EdgesAsList(pathB));
>       final := JoinVertexEdgePaths(join, holeB, fillB)[1];
>       return final;
> end;
```

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